

# Anomaly Cancellation in Field Theory and F-theory on a Circle

Thomas W. Grimm and Andreas Kapfer <sup>1</sup>

Max-Planck-Institut für Physik,  
Föhringer Ring 6, 80805 Munich, Germany

## ABSTRACT

We study the manifestation of local gauge anomalies of four- and six-dimensional field theories in the lower-dimensional Kaluza-Klein theory obtained after circle compactification. We identify a convenient set of transformations acting on the whole tower of massless and massive states and investigate their action on the low-energy effective theories in the Coulomb branch. The maps employ higher-dimensional large gauge transformations and precisely yield the anomaly cancellation conditions when acting on the one-loop induced Chern-Simons terms in the three- and five-dimensional effective theory. The arising symmetries are argued to play a key role in the study of the M-theory to F-theory limit on Calabi-Yau manifolds. For example, using the fact that all fully resolved F-theory geometries inducing multiple Abelian gauge groups or non-Abelian groups admit a certain set of symmetries, we are able to generally show the cancellation of pure Abelian or pure non-Abelian anomalies in these models.

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<sup>1</sup>grimm, kapfer@mpp.mpg.de

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# 1 Introduction

A study of the consistency of quantum field theories requires to investigate their local symmetries both at the classical and quantum level. In particular, even if such gauge symmetries are manifest in the classical theory, they might be broken at the quantum level and induce a violation of essential current conservation laws. Such inconsistencies manifest themselves already at one-loop level and are known as anomalies [1–3]. Four-dimensional quantum field theories with chiral spin- $1/2$  fermions, for example, can admit anomalies which signal the breaking of the gauge symmetry. Consistency requires the cancelation of these anomalies either by restricting the chiral spectrum such that a cancelation among various contributions takes place, or by implementing a generalized Green-Schwarz mechanism [4, 5]. The latter mechanism requires the presence of a  $U(1)$  gauged axion-like scalar with tree-level diagrams canceling the one-loop anomalies. In six space-time dimensions anomalies pose even stronger constraints, since in addition to spin- $1/2$  fermions also spin- $3/2$  and two-tensors can be chiral. Also in this case a generalized Green-Schwarz mechanism can be applied to cancel some of these anomalies.

In this work we address the manifestation of anomaly cancelation in four-dimensional and six-dimensional field theories from a Kaluza-Klein perspective when considering the theories to be compactified on a circle. Note that on a circle one can expand all higher-dimensional fields into Kaluza-Klein modes yielding a massless lowest mode and a tower of massive excited modes. Clearly, keeping track of this infinite set of fields one retains the full information about the higher-dimensional theory, including its anomalies. In a next step, one can compute the lower-dimensional effective theory for the massless modes only. This requires to integrate out all massive states. Of particular interest for the discussion of anomalies are the effective lower-dimensional couplings that are topological in nature. These do not continuously depend on the cutoff scale and therefore might receive quantum corrections from integrating out the massive states. Prominent examples are three-dimensional gauge Chern-Simons terms as well as five-dimensional gauge and gravitational Chern-Simons terms. These couplings are indeed modified at one-loop when integrating out massive states. In three dimensions only certain massive spin- $1/2$  fermions contribute [6–8], while in five dimensions also massive spin- $3/2$  and massive self-dual tensors give a non-vanishing shift [9–11]. In fact, precisely those modes contribute that arise from higher-dimensional chiral fields. Therefore, one expects that the Chern-Simons terms of the effective theories encode information about the higher-dimensional anomalies. This was recently investigated motivated by the study of F-theory effective actions via M-theory in [12–17]. With a different motivation similar questions were addressed in [18–25] in the study of applications of holography.

The connection between one-loop Chern-Simons terms and anomalies in the higher-dimensional theory, while expected to exist, was only shown to be rather indirect. In fact, it is not at all obvious how the anomaly cancelation conditions arise, for example, from comparing classical and one-loop Chern-Simons terms. While for many concrete examples in F-theory it was possible to check anomaly cancelation using the lower-dimensional

effective theory and Chern-Simons terms arising from M-theory, there was no known systematics behind this as of now. In this work we will suggest that there is an elegant way to actually approach this generally by describing symmetry transformations among effective theories that exist if higher-dimensional anomalies are canceled.

Let us consider an effective theory obtained after circle reduction. If the higher-dimensional theory admits a gauge group one can use the Wilson-line scalars of the gauge fields around the circle to move to the lower-dimensional Coulomb branch. In other words, one considers situations in which these Wilson line scalars admit a vacuum expectation value, which we call Coulomb branch parameters in the following. The masses of all the massive states are now dependent both on the circle radius, if they are excited Kaluza-Klein states, and on the Coulomb branch parameters, if they were charged under the higher-dimensional gauge group. With this in mind one can then compute the effective theory for the massless modes and focus on the Chern-Simons terms. While the one-loop Chern-Simons terms are not continuous functions of the masses of the integrated-out states, they can experience discrete shifts, when changing the radius or the Coulomb branch parameters. In other words, depending on the background value of the Coulomb branch parameters and the radius, the effective theories can take a different form. A priori, one would thus expect that one finds infinitely many values for the Chern-Simons coefficient due to the infinitely many hierarchies of Kaluza-Klein masses and Coulomb branch masses. However, we argue that there are certain transformations arising from higher-dimensional large gauge transformations that identify different Coulomb branch parameters and effective theories if and only if anomalies are canceled. Importantly, the transformations are designed to yield the anomaly cancellation conditions when considering the classical and one-loop Chern-Simons terms of the effective theory. Our goal is to examine these maps both for four-dimensional and six-dimensional gauge theories with a focus on pure non-Abelian and pure Abelian anomalies.

While most of our discussion is purely field-theoretic, it is important to stress that the original motivation to carry out such a study arose from the analysis of anomaly cancellation in F-theory via the M-theory dual. Recall that F-theory on a complex four- or three-dimensional manifold yields a four-dimensional and six-dimensional effective theory, respectively [26–31, 14, 15]. The geometry of the internal manifold dictates both the gauge group and matter content that arises from space-time filling seven-branes. To study four- and six-dimensional field theories that arise from F-theory, however, one needs to take a detour via M-theory. In fact, one can derive an effective theory in three or five dimensions by starting with eleven-dimensional supergravity dimensionally reduced on the completely resolved geometry. Different gauge theory phases of such theories and their relation to geometric resolutions have been recently studied in [32, 12, 33–37]. These effective theories are in the lower-dimensional Coulomb branch and dual to the F-theory effective actions on an extra circle with all massive modes integrated out. Therefore, one is precisely in the situation we consider in purely field-theoretic terms.

The transformations we consider in order to map effective theories descend from transformations acting on the resolved F-theory geometries. This implies that if the trans-

formations are in fact symmetries of the geometries and the M-theory to F-theory limit, then anomalies are canceled. Indeed we are able to show that in purely Abelian theories the transformations on the geometry correspond to *picking a zero-section*, i.e. identifying the Kaluza-Klein vector of the F-theory circle compactification in the internal geometry. Since nothing dictates a preferred choice of zero-section this has to be a symmetry of the M-theory to F-theory limit. This shows that anomalies for purely Abelian theories are generally canceled for the considered F-theory geometries. In order to give a proof of generic cancelation of non-Abelian anomalies we need to identify the corresponding geometrical symmetry. Remarkably, we find that it corresponds to ‘*picking a zero-node*’, which was as of now always done by a canonical choice using the zero-section. Having established the presence of this extra symmetry we argue for the generic cancelation of pure non-Abelian anomalies in the considered F-theory geometries.

This paper is organized as follows. In section 2 and section 3 we perform the field-theoretic analysis of anomalies in four and six dimensions, respectively. In both dimensions we first focus on purely non-Abelian models and later on purely Abelian models. We give a detailed account of the transformations that become symmetries among effective theories once anomalies are canceled. In section 4 we turn to the analysis of the F-theory geometries and make extensive use of the duality between M-theory and F-theory on an extra circle. Identifying actual symmetries of the geometries we are able to show the general cancelation of pure Abelian and pure non-Abelian anomalies. We supplement additional information in four appendices. Relevant identities valid on the Coulomb branch are discussed in Appendix A, while details on the one-loop computations are summarized in Appendix B. Useful group theory identities are summarized in Appendix C, where we also translate trace identities into relations among weights. Some important results on the intersection numbers of our geometries are given in Appendix D.

## 2 Anomalies of four-dimensional theories

In this section we study anomaly cancelation of four-dimensional matter-coupled gauge theories from the three-dimensional Kaluza-Klein perspective of the circle compactified theories on the Coulomb branch. More precisely, we analyze the Chern-Simons terms of classes of three-dimensional effective theories that arise at different values of the Coulomb branch parameters after integrating out all massive modes. After introducing the generalities on the four-dimensional setup in subsection 2.1, we discuss the circle reduction and one-loop Chern-Simons terms in subsection 2.2. If anomalies are canceled, a set of transformations induced by a higher-dimensional large gauge transformation identifies infinitely many effective theories and thus represents a symmetry. The motivation to study these transformation arose originally from the duality of M-theory to F-theory, as detailed in section 4, but applies to arbitrary matter-coupled gauge theories in four dimensions, including also a possible coupling to gravity. For simplicity we will only investigate the two cases of a pure non-Abelian and pure Abelian gauge group in subsection 2.3 and subsection 2.4, respectively. We are confident that the reasoning generalizes

also to reductive gauge groups.

## 2.1 General setup and anomaly conditions

Let us first introduce the general setup before restricting to pure non-Abelian and pure Abelian gauge groups, respectively. We use  $G$  to denote a simple gauge group<sup>2</sup> with gauge bosons  $\hat{A}$ . Denoting by  $T_{\mathcal{I}}$ ,  $\mathcal{I} = 1, \dots, \dim G$  the Lie algebra generators, we expand

$$\hat{A} = \hat{A}^{\mathcal{I}} T_{\mathcal{I}} = \hat{A}^I T_I + \hat{A}^{\alpha} T_{\alpha} \quad (2.1)$$

where  $T_I$ ,  $I = 1, \dots, \text{rank } G$  are the generators of the Cartan subalgebra and  $T_{\alpha}$  are the remaining generators labeled by the roots  $\alpha$ . Our conventions in the theory of Lie algebras and their representations are listed in Appendix C. The  $n_{U(1)}$  Abelian gauge bosons are denoted by  $\hat{A}^m$  with  $m = 1, \dots, n_{U(1)}$ .

Our setup also includes charged matter and we focus on chiral spin-1/2 Weyl fermions since they are the only states contributing gauge anomalies. We write  $F(R)$  for the chiral index of a representation  $R$  of these fermions  $\hat{\psi}$  under the gauge group  $G$ . Their covariant derivative reads

$$\mathcal{D}_{\mu} \hat{\psi} = \nabla_{\mu} \hat{\psi} - i \hat{A}_{\mu}^{\mathcal{I}} T_{\mathcal{I}}^R \hat{\psi}, \quad (2.2)$$

where  $T_{\mathcal{I}}^R$  are the Lie algebra generators in the representation  $R$ . Upon choosing an eigenbasis associated to the weights  $w$  of  $R$  and expanding  $\hat{\psi}$  with coefficients  $\hat{\psi}(w)$  one finds for the Cartan directions

$$T_I^R \hat{\psi}(w) = w_I \hat{\psi}(w), \quad w_I := \langle \alpha_I^{\vee}, w \rangle, \quad (2.3)$$

where  $\alpha_I^{\vee}$  is the simple coroot associated to  $T_I$ . Similarly,  $F(q)$  is the chiral index of spin-1/2 Weyl fermions  $\hat{\psi}$  with charges  $q = (q_m)$  under  $\hat{A}^m$  encoded in the covariant derivative

$$\mathcal{D}_{\mu} \hat{\psi} = \nabla_{\mu} \hat{\psi} - i q_m \hat{A}_{\mu}^m \hat{\psi}. \quad (2.4)$$

Moreover, we restrict to fermions without four-dimensional mass terms. Note that since we treat non-Abelian and Abelian theories separately, we do not need fermions that are charged both under  $\hat{A}$  and  $\hat{A}^m$ .

In order to cancel anomalies a four-dimensional Green-Schwarz mechanism might be required [4, 1–3]. Therefore, we also allow for a number of  $n_{\text{ax}}$  axions  $\hat{\rho}_{\alpha}$ ,  $\alpha = 1, \dots, n_{\text{ax}}$  with covariant derivative

$$\mathcal{D} \hat{\rho}_{\alpha} = d \hat{\rho}_{\alpha} + \theta_{\alpha m} \hat{A}^m \quad (2.5)$$

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<sup>2</sup>The generalization to semi-simple gauge groups is straightforward.

with  $\theta_{\alpha m}$  constant. The possibly gauge non-invariant couplings of the axions to the gauge fields read

$$\hat{S}_{\text{GS}} = \int -\frac{1}{4} b_{mn}^{\alpha} \hat{\rho}_{\alpha} \hat{F}^m \wedge \hat{F}^n - \frac{1}{4 \lambda(G)} b^{\alpha} \hat{\rho}_{\alpha} \text{tr}_f \hat{F} \wedge \hat{F}, \quad (2.6)$$

where  $b_{mn}^{\alpha}$ ,  $b^{\alpha}$  are the Green-Schwarz coefficients, the trace  $\text{tr}_f$  is in the fundamental representation of  $G$ , and  $\lambda(G)$  are normalization factors discussed in Appendix C.<sup>3</sup> We stress that our considerations work both in theories with or without an implemented Green-Schwarz mechanism. Furthermore, it is of course possible to include other non-chiral or uncharged fields, since these contribute neither to the anomalies nor to the one-loop Chern-Simons terms of the circle-reduced theory discussed in subsection 2.2. Finally, it is also possible to couple the theory to gravity. The arguments concerning the gauge anomalies will be unaltered by this generalization.

In subsection 2.3 and subsection 2.4 we will see that via a certain basis transformation of the vectors one can recover the four-dimensional gauge anomaly equations from one-loop Chern-Simons terms in three dimensions. Let us therefore display the purely non-Abelian and purely Abelian anomaly conditions in four dimensions<sup>4</sup>

$$\sum_R F(R) V_R = 0, \quad (2.7a)$$

$$\sum_q F(q) q_m q_n q_p = \frac{3}{2} b_{(mn}^{\alpha} \theta_{p)\alpha}, \quad (2.7b)$$

where  $V_R$  is defined as

$$\text{tr}_R \hat{F}^3 = V_R \text{tr}_f \hat{F}^3. \quad (2.8)$$

Here  $\text{tr}_R$ ,  $\text{tr}_f$  are the traces in the representation  $R$  and the fundamental representation respectively. We have now introduced all relevant parts of the field theory setup that are necessary for our considerations.

## 2.2 Circle compactification and Chern-Simons terms

In the next step we compactify this theory on a circle and push it to the Coulomb branch by allowing for a non-vanishing Wilson line background of the gauge fields reduced on the circle.

We indicate four-dimensional quantities by a hat, while three-dimensional ones lack a hat. At lowest Kaluza-Klein (KK) level we get  $\dim G$   $U(1)$  gauge fields  $A^T$  and  $\dim G$

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<sup>3</sup>In gravity theories there may be an additional Green-Schwarz coupling in order to cancel mixed gravitational anomalies. Since we only treat gauge anomalies in four dimensions, we will omit this term in the following. Also the Green-Schwarz coupling for the non-Abelian gauge fields  $b^{\alpha}$  only plays a role in the cancelation of mixed Abelian-non-Abelian anomalies that are not discussed in this paper. We nevertheless list it here because it will appear when treating the M-/F-theory duality in section 4.

<sup>4</sup>All symmetrizations over  $n$  indices include a factor of  $\frac{1}{n!}$ .

Wilson line scalars  $\zeta^{\mathcal{I}}$  from reducing  $\hat{A}$ . In the Abelian sector we obtain  $n_{U(1)}$   $U(1)$  gauge fields  $A^m$  and Wilson line scalars  $\zeta^m$  from reducing  $\hat{A}^m$ . Furthermore, there are  $n_{\text{ax}}$  Abelian gauge fields  $A^\alpha$  from dualizing the reduced  $\hat{\rho}^\alpha$ . From the four-dimensional metric one finds at lowest level the three-dimensional metric  $g_{\mu\nu}$ , the KK-vector  $A^0$  and the radius  $r$  of the circle. More precisely, we expand the four-dimensional metric as

$$d\hat{s}^2 = g_{\mu\nu}dx^\mu dx^\nu + r^2 Dy^2, \quad Dy := dy - A_\mu^0 dx^\mu, \quad (2.9)$$

with  $x^\mu$  the three-dimensional coordinates and  $y$  the coordinate along the circle. The three-dimensional metric  $g_{\mu\nu}$  is taken to be that of Minkowski space for simplicity. The gauge fields are expanded according to

$$\hat{A}^{\mathcal{I}} = A^{\mathcal{I}} - \zeta^{\mathcal{I}} r Dy, \quad \hat{A}^m = A^m - \zeta^m r Dy. \quad (2.10)$$

Note that the  $\zeta^{\mathcal{I}}$  are transforming in the adjoint representation of  $G$ .

The massive fields in the theory are the KK-modes and states that acquire masses on the Coulomb branch. The three-dimensional Coulomb branch is parametrized by the background values of the scalars  $\zeta^{\mathcal{I}}$  and  $\zeta^m$ , by setting

$$\langle \zeta^I \rangle \neq 0, \quad \langle \zeta^\alpha \rangle = 0, \quad \langle \zeta^m \rangle \neq 0, \quad (2.11)$$

i.e. giving the Cartan Wilson line scalars a vacuum expectation value (VEV). This breaks  $G \times U(1)^{n_{U(1)}} \rightarrow U(1)^{\text{rank } G} \times U(1)^{n_{U(1)}}$  and gives the W-bosons  $A^\alpha$  a mass. Also the modes of the higher-dimensional charged matter states will gain a mass. To find the Coulomb branch masses  $m_{\text{CB}}^w$  for a state  $\psi(w)$  labeled with a weight  $w$ , or a state  $\psi(q)$  with charge  $q$ , we use (2.2)-(2.4) to read off

$$m_{\text{CB}}^w = w_I \langle \zeta^I \rangle, \quad m_{\text{CB}}^q = q_m \langle \zeta^m \rangle, \quad (2.12)$$

In total the mass of a field  $\psi_{(n)}(w)$  at KK-level  $n$  in the three-dimensional theory reads

$$m = \begin{cases} m_{\text{CB}}^w + n m_{\text{KK}} = w_I \langle \zeta^I \rangle + \frac{n}{\langle r \rangle}, & \text{non-Abelian gauge group,} \\ m_{\text{CB}}^q + n m_{\text{KK}} = q_m \langle \zeta^m \rangle + \frac{n}{\langle r \rangle}, & \text{Abelian gauge group,} \end{cases} \quad (2.13)$$

with  $m_{\text{KK}} = 1/\langle r \rangle$  being the unit KK-mass determined by the background value of the radius.

Of key importance in this paper are the three-dimensional Chern-Simons terms. For a general Abelian theory they take the form

$$S_{\text{CS}} = \int \Theta_{\Lambda\Sigma} A^\Lambda \wedge F^\Sigma, \quad (2.14)$$

where  $\Theta_{\Lambda\Sigma}$  are constants. We introduced the collective index  $\Lambda = (0, I, \alpha)$  or  $\Lambda = (0, m, \alpha)$ , respectively. Performing a classical Kaluza-Klein reduction we straightforwardly find the Chern-Simons coefficients

$$\Theta_{\alpha\beta} = 0, \quad \Theta_{\alpha 0} = 0, \quad \Theta_{\alpha I} = 0, \quad \Theta_{\alpha m} = \frac{1}{2} \theta_{\alpha m}. \quad (2.15)$$



Note that the non-zero coefficient descends from the Green-Schwarz couplings in four dimensions.

We are interested in the three-dimensional effective theory for the massless modes only. This implies that all massive states need to be integrated out. Importantly, one thus needs to include one-loop corrections to the Chern-Simons terms. It is well-known that a massive charged spin- $1/2$ -fermion contributes to  $\Theta_{\Lambda\Sigma}$  as [6–8]

$$\Theta_{\Lambda\Sigma}^{\text{loop}} = \frac{1}{2} q_{\Lambda} q_{\Sigma} \text{sign}(m), \quad (2.16)$$

where  $q_{\Lambda}$  is the charge of the fermion under the  $U(1)$  gauge boson  $A^{\Lambda}$  and the sign of the mass encodes the spinor representation of the fermion. We adopt the convention that a KK-mode  $\psi_{(n)}$  is charged under the KK-vector in the following way

$$D_{\mu}\psi_{(n)} = \partial_{\mu}\psi_{(n)} + inA_{\mu}^0\psi_{(n)} \quad (2.17)$$

This means that the charge of  $\psi_{(n)}$  under  $A^0$  is  $q_0 = -n$ .

Summing up all contributions to the one-loop Chern-Simons terms in the circle-reduced theory requires to include the infinite KK-tower that needs to be treated with zeta function regularization. The relevant computations are carried out in Appendix B. For the present setup the relevant total one-loop Chern-Simons coefficients for the pure non-Abelian and the pure Abelian theory respectively are evaluated as [15, 16]

$$\Theta_{IJ} = \sum_R F(R) \sum_{w \in R} \left(l_w + \frac{1}{2}\right) w_I w_J \text{sign}(m_{\text{CB}}^w), \quad (2.18a)$$

$$\Theta_{mn} = \sum_q F(q) \left(l_q + \frac{1}{2}\right) q_m q_n \text{sign}(m_{\text{CB}}^q), \quad (2.18b)$$

where the sums in the first equation are over all representations  $R$  of  $G$  and all weights of a given representation. We stress that in all sums over weights in this paper the multiplicity factors of the weights are always meant to be implicitly included into the sum. The integers  $l_w, l_q$  are defined as

$$l_w = \left\lfloor \left| \frac{m_{\text{CB}}^w}{m_{\text{KK}}} \right| \right\rfloor, \quad l_q = \left\lfloor \left| \frac{m_{\text{CB}}^q}{m_{\text{KK}}} \right| \right\rfloor, \quad (2.19)$$

where the brackets indicate the use of the floor function. The  $l_w, l_q$  indicate at which KK-level the sign of the total mass of a state becomes independent of the Coulomb branch mass. The remaining one-loop Chern-Simons coefficients are listed for completeness in Appendix B. We stress that the classical Chern-Simons coefficients (2.15) receive no corrections at one-loop.

Having introduced the relevant parts of the circle-reduced effective theory, we are now in a position to show in detail how gauge anomalies in four dimensions are related to symmetries on the Coulomb branch in three dimensions.

## 2.3 Non-Abelian models

In this section we drop the Abelian gauge bosons  $\hat{A}^m$  and restrict to a pure non-Abelian simple gauge group  $G$ . We consider a certain basis transformation of the vectors in the circle reduced theory.<sup>5</sup> Firstly, choose an arbitrary gauge field  $A^{\tilde{0}}$  out of the Cartan vectors  $A^I$ . The remaining Cartan vectors are denoted by  $A^{\tilde{I}}$ , while the remaining gauge fields of the whole set  $A^{\mathcal{I}}$ ,  $\mathcal{I} = 1, \dots, \dim G$  will be denoted by  $A^{\tilde{\mathcal{I}}}$ . Then we define the transformation

$$\begin{pmatrix} \tilde{A}^0 \\ \tilde{A}^{\tilde{0}} \\ \tilde{A}^{\tilde{\mathcal{I}}} \\ \tilde{A}^\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \delta_{\tilde{\mathcal{J}}}^{\tilde{\mathcal{I}}} & 0 \\ 0 & 0 & 0 & \delta_\beta^\alpha \end{pmatrix} \cdot \begin{pmatrix} A^0 \\ A^{\tilde{0}} \\ A^{\tilde{\mathcal{J}}} \\ A^\beta \end{pmatrix}. \quad (2.20)$$

All quantities in the transformed basis are labeled by a tilde. In the following we again collectively denote the fields  $\tilde{A}^{\tilde{0}}, \tilde{A}^{\tilde{\mathcal{I}}}$  by  $\tilde{A}^{\tilde{\mathcal{I}}}$ . It is essential to notice that the KK-vector mixes with the Cartan  $U(1)$  gauge field  $A^{\tilde{0}}$ , since this fact will render the basis transformation non-trivial. This implies that, as long as one keeps all gauge fields including the W-bosons, the non-Abelian gauge transformations are realized on the tilted basis in a very non-trivial fashion. It is also worthwhile to note that the map leaves the classical Chern-Simons terms (2.15) invariant.

The transformation (2.20) also requires to transform the charged fields in the KK-theory. Given a matter state  $\psi_{(n)}$  at KK-level  $n$  in the representation  $R$  of  $G$  one first chooses a basis associated to the weights writing  $\psi_{(n)}(w)$  as in (2.3). The transformation (2.20) mixes these states as

$$\psi_{(n)}(w) \rightarrow \psi_{(\tilde{n})}(\tilde{w}), \quad \begin{pmatrix} \tilde{n} \\ \tilde{w}_{\tilde{0}} \\ \tilde{w}_{\tilde{I}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} n \\ w_{\tilde{0}} \\ w_{\tilde{I}} \end{pmatrix}. \quad (2.21)$$

Note that in general this transformation shifts the whole KK-tower, but there is still no state charged under  $\tilde{A}^\alpha$ . As we will see momentarily, the shift (2.21) has important consequences due to the fact that the contributions of the infinite KK-tower to the Chern-Simons terms have to be regularized.

Let us now investigate the impact of the transformations (2.20) and (2.21) on the effective theories for the massless fields only. We stress that there is not only one effective theory, but rather an infinite set of such theories labeled by the vacuum expectation values  $\langle \zeta^I \rangle$ ,  $\langle r \rangle$  that control the masses of the states. The effective theories are distinguished, in particular, through their one-loop Chern-Simons terms (2.18a) and (2.18b), which

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<sup>5</sup> This transformation is inspired by the M-/F-theory duality, as explained in section 4, but our analysis is purely field-theoretic in the following.

change when changing  $\langle \zeta^I \rangle$ ,  $\langle r \rangle$ . Note that compatibility with (2.10) requires that the transformation (2.20) is accompanied by a shift of these vacuum expectation values as

$$\frac{1}{\langle r \rangle} \mapsto \frac{1}{\langle r \rangle}, \quad \langle \zeta^{\tilde{I}} \rangle \mapsto \langle \zeta^{\tilde{I}} \rangle, \quad \langle \zeta^{\tilde{0}} \rangle \mapsto \langle \zeta^{\tilde{0}} \rangle - \frac{1}{\langle r \rangle}. \quad (2.22)$$

In other words, such a transformation in general maps one effective theory to a different effective theory. However, note that it is not hard to check that (2.22) can be understood as a large gauge transformation in the underlying four-dimensional theory. Of course, if the theory is gauge-invariant the two effective theories related by this transformation need to be identical. Gauge-invariance is here tested at the loop level since we consider one-loop Chern-Simons terms. Indeed, we will show in the following that the transformations (2.20), (2.21), and (2.22) identify two three-dimensional theories with distinct Coulomb branch parameters but equivalent effective theories, if and only if four-dimensional anomalies are canceled.

To make this more precise we first perform the map (2.20) to the tilted basis. In this transformed basis we directly evaluate the one-loop Chern-Simons coefficient  $\tilde{\Theta}_{IJ}$  using (2.18a) as

$$\tilde{\Theta}_{IJ} = \sum_R F(R) \sum_{w \in R} w_I w_J \left( \tilde{l}_w + \frac{1}{2} \right) \text{sign}(\tilde{m}_{\text{CB}}^w); \quad (2.23)$$

note that  $\tilde{w}_I = w_I$ . Alternatively we can transform  $\tilde{\Theta}_{IJ}$  back to the old basis and perform the loop-calculation there

$$\tilde{\Theta}_{IJ} = \Theta_{IJ} = \sum_R F(R) \sum_{w \in R} w_I w_J \left( l_w + \frac{1}{2} \right) \text{sign}(m_{\text{CB}}^w). \quad (2.24)$$

We can now match (2.23) with (2.24) and use the identity

$$w_{\tilde{0}} = \left( l_w + \frac{1}{2} \right) \text{sign}(m_{\text{CB}}^w) - \left( \tilde{l}_w + \frac{1}{2} \right) \text{sign}(\tilde{m}_{\text{CB}}^w), \quad (2.25)$$

which is proven in Appendix A. This yields the equation

$$\sum_R F(R) \sum_{w \in R} w_I w_J w_{\tilde{0}} = 0, \quad (2.26)$$

which not only implies the pure non-Abelian anomaly (2.7a) but is actually equivalent to it, since for some combinations of indices already the sum  $\sum_{w \in R} w_I w_J w_{\tilde{0}}$  is trivially zero as a group theoretical consequence. This is shown in Appendix C. Performing this matching for the other one-loop Chern-Simons coefficients  $\tilde{\Theta}_{00}$ ,  $\tilde{\Theta}_{0I}$  and picking different gauge fields for the distinguished field  $A^{\tilde{0}}$  one again obtains either the anomaly condition or trivially zero as consequence of representation theory.

Finally, let us comment on the group structure of the transformations (2.20). It turns out that the transformations for different choices of  $A^{\tilde{0}}$  all commute. We conclude that

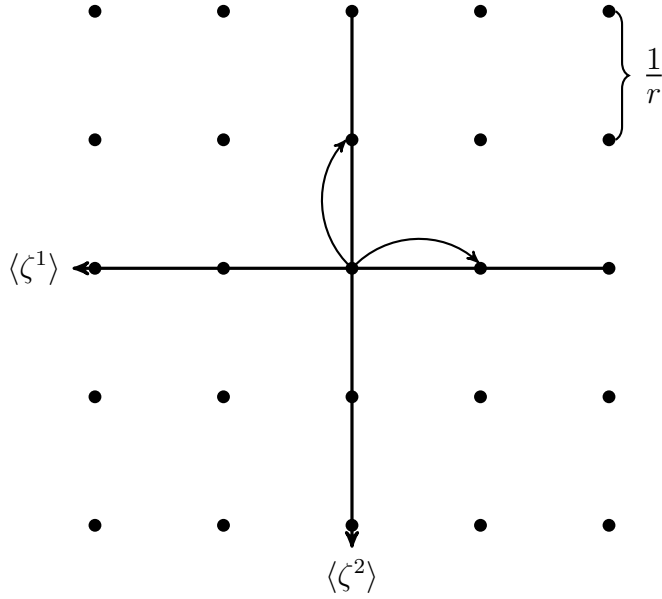


Figure 1: The transformations (2.20) generate a  $\mathbb{Z}^{\text{rank } G}$  group relating theories with different Coulomb branch parameters but the same effective theory, as long as anomalies are canceled in the four-dimensional theory. We display the lattice of related theories for rank  $G = 2$ . The transformation associated to the horizontal arrow corresponds to the choice  $A^1 \equiv A^{\tilde{0}}$ , the other one to  $A^2 \equiv A^{\tilde{0}}$ .

the maps (2.20) generate a group  $\mathbb{Z}^{\text{rank } G}$ . It relates infinitely many classes of theories on the Coulomb branch that all lead to the same effective physics if and only if anomalies are absent in four dimensions. This is expected when noting that (2.20) and (2.22) can be associated to a higher-dimensional large gauge transformation. In fact, in an anomaly-free theory one therefore finds that the moduli space of the Coulomb branch VEVs  $\langle \zeta^I \rangle$  is actually found to be a torus.<sup>6</sup> For a gauge group with rank two this is illustrated in Figure 1.

## 2.4 Abelian models

Let us now move to the purely Abelian theory, i.e. we drop the fields  $\hat{A}$  and include  $\hat{A}^m$ . In contrast to the previous section we now face a lot more structure because there may now be a Green-Schwarz mechanism at work. Similar to the preceding discussion we consider a basis transformation of the vectors, for which we first choose an arbitrary gauge field  $A^{\tilde{0}}$  out of the  $A^m$ . The remaining gauge fields of  $A^m$  will be denoted by  $A^{\tilde{m}}$ .

<sup>6</sup>We like to thank Federico Bonetti for discussions on these points.

The map is then defined as

$$\begin{pmatrix} \tilde{A}^0 \\ \tilde{A}^{\tilde{0}} \\ \tilde{A}^{\tilde{m}} \\ \tilde{A}^\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \delta_{\tilde{n}}^{\tilde{m}} & 0 \\ \frac{1}{2}b_{00}^\alpha & b_{00}^\alpha & b_{0\tilde{n}}^\alpha & \delta_\beta^\alpha \end{pmatrix} \cdot \begin{pmatrix} A^0 \\ A^{\tilde{0}} \\ A^{\tilde{n}} \\ A^\beta \end{pmatrix}. \quad (2.27)$$

The transformed quantities carry a tilde and  $\tilde{A}^{\tilde{0}}, \tilde{A}^{\tilde{m}}$  are again collected in  $\tilde{A}^m$ . The group structure is in analogy to the last subsection  $\mathbb{Z}^{n_{U(1)}}$ . Again one expects that (2.27) can be associated to a large gauge transformation in the four-dimensional theory and it would be nice to check this explicitly.

This time the classical Chern-Simons terms (2.15) are all invariant except of

$$\Theta_{\alpha 0} \mapsto \tilde{\Theta}_{\alpha 0} = \Theta_{\alpha \tilde{0}}. \quad (2.28)$$

Recall that in the circle reduced theory we originally had  $\Theta_{\alpha 0} = 0$ , but  $\Theta_{\alpha \tilde{0}} \neq 0$  as in (2.15) in the presence of a Green-Schwarz mechanism. Therefore  $\tilde{\Theta}_{\alpha 0} \neq \Theta_{\alpha 0}$ , which contradicts our requirement for the classical Chern-Simons coefficients to be invariant. This issue can be cured by performing additional shifts  $\delta\tilde{\Theta}_{\Lambda\Sigma}$  for the Chern-Simons coefficients  $\tilde{\Theta}_{\Lambda\Sigma}$ .<sup>7</sup> Explicitly, they read

$$\delta\tilde{\Theta}_{\alpha 0} = -\Theta_{\alpha \tilde{0}}, \quad \delta\tilde{\Theta}_{mn} = b_{mn}^\alpha \Theta_{\alpha \tilde{0}}. \quad (2.29)$$

All other Chern-Simons coefficients remain unchanged. At first, these shifts appear mysterious even though they are straightforwardly motivated in the M-/F-theory setup. A field theory explanation exploits the fact that the coefficients  $\Theta_{\alpha 0}$  and  $\Theta_{mn}$  arise *classically* if one performs a KK-reduction with circle fluxes for the axions  $\hat{\rho}_\alpha$

$$\Theta_{\alpha 0} = \frac{1}{2} \int_{S^1} \langle d\hat{\rho}_\alpha \rangle, \quad (2.30a)$$

$$\Theta_{mn}^{\text{class}} = -\frac{1}{2} b_{mn}^\alpha \int_{S^1} \langle d\hat{\rho}_\alpha \rangle. \quad (2.30b)$$

We thus realize that large gauge transformations in this case induce circle fluxes for axions. The shifts (2.29) undo this modification by manually switching on compensating flux in the other direction along the circle. As we will see in the next section, these shifts are not necessary in the six-dimensional setup.

Finally, let us determine how (2.27) acts on the charges of the fields in the theory. For states  $\psi_{(n)}(q)$  we find that they mix according to

$$\begin{pmatrix} \tilde{n} \\ \tilde{q}_{\tilde{0}} \\ \tilde{q}_{\tilde{m}} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} n \\ q_{\tilde{0}} \\ q_{\tilde{m}} \end{pmatrix}. \quad (2.31)$$

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<sup>7</sup> Again this can be motivated by the M-/F-theory realization of this transformation. In section 4 we show that it is related to the transformation of  $G_4$ -flux under (2.27).

As before we can apply this transformation to the calculation of one-loop Chern-Simons terms.

The one-loop Chern-Simons coefficient  $\tilde{\Theta}_{mn}$  can be directly evaluated in the transformed basis using (2.18b)

$$\tilde{\Theta}_{mn} = \sum_q F(q) q_m q_n \left( \tilde{l}_q + \frac{1}{2} \right) \text{sign}(\tilde{m}_{\text{CB}}^q), \quad (2.32)$$

where we used  $\tilde{q}_m = q_m$ . Mapping  $\tilde{\Theta}_{mn}$  back to the old basis and calculating the loop there we obtain

$$\begin{aligned} \tilde{\Theta}_{mn} &= \Theta_{mn} - b_{0m}^\alpha \Theta_{\alpha n} - b_{0n}^\alpha \Theta_{\alpha m} \\ &= \sum_q F(q) q_m q_n \left( l_q + \frac{1}{2} \right) \text{sign}(m_{\text{CB}}^q) - \frac{1}{2} b_{0m}^\alpha \theta_{\alpha n} - \frac{1}{2} b_{0n}^\alpha \theta_{\alpha m}. \end{aligned} \quad (2.33)$$

We can now match (2.32) with (2.33) supplemented by the shift (2.29) and use the identity

$$q_{\tilde{0}} = \left( l_q + \frac{1}{2} \right) \text{sign}(m_{\text{CB}}^q) - \left( \tilde{l}_q + \frac{1}{2} \right) \text{sign}(\tilde{m}_{\text{CB}}^q), \quad (2.34)$$

which is proven in Appendix A. We end up with the equation

$$\sum_q F(q) q_m q_n q_{\tilde{0}} = \frac{3}{2} b_{(mn)\tilde{0}}^\alpha \theta_{\alpha} \quad (2.35)$$

and recognize it as a subset of the purely Abelian anomaly equations in four dimensions (2.7b). Performing this matching for the other one-loop Chern-Simons coefficients  $\tilde{\Theta}_{00}$ ,  $\tilde{\Theta}_{0m}$  and picking different gauge fields for the distinguished field  $A^{\tilde{0}}$  in the transformation one is able to generate the full set of Abelian anomaly equations (with repetitions).

### 3 Anomalies of six-dimensional theories

The techniques to obtain the anomaly equations via circle reduction developed in the previous section work similarly in six dimensions. The generalities on the six-dimensional setup and its anomaly cancelation conditions are introduced in subsection 3.1. This theory is compactified on a circle in subsection 3.2, where we also recall the form of the five-dimensional Chern-Simons terms arising by integrating out massive modes. The Kaluza-Klein perspective on the anomaly cancelation conditions for non-Abelian and Abelian gauge groups is discussed in subsection 3.3 and subsection 3.4, respectively. In addition to the pure gauge anomalies we are able to derive the mixed gauge-gravitational anomaly equation from considering higher-curvature Chern-Simons terms.

### 3.1 General setup and anomaly conditions

To simplify the discussion we analyze the two cases of a pure non-Abelian simple gauge group  $G$  and a pure Abelian gauge group in detail. We start first by introducing the general setup. The gauge bosons of the simple group  $G$  are denoted by  $\hat{A}$  with components  $\hat{A}^{\mathcal{I}}$ ,  $\mathcal{I} = 1, \dots, \dim G$  and Cartan directions  $\hat{A}^I$ ,  $I = 1, \dots, \text{rank } G$ . The  $n_{U(1)}$  Abelian gauge fields are denoted by  $\hat{A}^m$  with  $m = 1, \dots, n_{U(1)}$ .

We introduce the chiral index  $F_{1/2}(R)$  of representations  $R$  of spin- $1/2$  Weyl spinors  $\hat{\psi}$  under  $G$ . The covariant derivative is given by

$$\mathcal{D}_\mu \hat{\psi} = \nabla_\mu \hat{\psi} - i \hat{A}_\mu^{\mathcal{I}} T_{\mathcal{I}}^R \hat{\psi}, \quad (3.1)$$

where  $T_{\mathcal{I}}^R$  acts in the representation  $R$  as discussed above. Furthermore  $F_{1/2}(q)$  is the chiral index of spin- $1/2$  Weyl spinors  $\hat{\psi}$  with charges  $q = (q_m)$ , whose covariant derivative reads

$$\mathcal{D}_\mu \hat{\psi} = \nabla_\mu \hat{\psi} - i q_m \hat{A}_\mu^m \hat{\psi}. \quad (3.2)$$

In six dimensions spin- $3/2$  Weyl spinors, as they e.g. appear in supergravity multiplets, can be chiral with chiral index  $F_{3/2}$  and therefore may be included in the anomaly analysis. We assume that these fermions are uncharged and therefore do not contribute to anomalies involving the gauge fields. Furthermore, a six-dimensional theory can admit  $T_{\text{sd}}$  self-dual and  $T_{\text{asd}}$  anti-self-dual two-forms  $\hat{B}_\alpha$ ,  $\alpha = 1, \dots, T_{\text{sd}} + T_{\text{asd}}$ . These forms carry a definite chirality and can contribute to the anomalies. We require that the tensors are not charged, however, they can couple with Green-Schwarz couplings to the gauge fields and curvature two-form  $\hat{\mathcal{R}}$  as

$$\hat{S}_{\text{GS}} = -\frac{1}{2} \int_{M_6} \Omega_{\alpha\beta} \hat{B}^\alpha \wedge \left( \frac{1}{2} a^\alpha \text{tr } \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} + 2 b_{mn}^\alpha \hat{F}^m \wedge \hat{F}^n + 2 \frac{b^\alpha}{\lambda(G)} \text{tr}_f \hat{F} \wedge \hat{F} \right), \quad (3.3)$$

where  $\lambda(G)$  is defined in Appendix C. The coefficients  $a^\alpha$ ,  $b_{mn}^\alpha$ ,  $b^\alpha$  are the Green-Schwarz coefficients, while  $\Omega_{\alpha\beta}$  is constant, symmetric in its indices and its signature consists of  $T_{\text{sd}}$  positive signs and  $T_{\text{asd}}$  negative ones. We stress that our considerations in the following subsections also apply to situations where no Green-Schwarz mechanism is applied to cancel anomalies. Also the coupling to gravity can be analyzed independently of the gauge theory analysis. Note that also other non-chiral fields may be present. Since they neither contribute to the anomaly nor do their descendants in five dimensions induce one-loop Chern-Simons terms, they have no effect in the following discussion.

Since we will reproduce six-dimensional gauge anomalies and mixed gravitational anomalies from loop-corrections in five dimensions, let us display these cancelation con-

ditions explicitly<sup>8</sup>

$$6a^\alpha \frac{b^\beta}{\lambda(G)} \Omega_{\alpha\beta} = \sum_R F_{1/2}(R) A_R, \quad (3.4a)$$

$$6a^\alpha b_{mn}^\beta \Omega_{\alpha\beta} = \sum_q F_{1/2}(q) q_m q_n, \quad (3.4b)$$

$$0 = \sum_R F_{1/2}(R) B_R, \quad (3.4c)$$

$$-3 \frac{b^\alpha}{\lambda(G)} \frac{b^\beta}{\lambda(G)} \Omega_{\alpha\beta} = \sum_R F_{1/2}(R) C_R, \quad (3.4d)$$

$$-(b_{mn}^\alpha b_{pq}^\beta + b_{mp}^\alpha b_{nq}^\beta + b_{mq}^\alpha b_{np}^\beta) \Omega_{\alpha\beta} = \sum_q F_{1/2}(q) q_m q_n q_p q_q, \quad (3.4e)$$

where the constants  $A_R$ ,  $B_R$ ,  $C_R$  are defined as

$$\begin{aligned} \text{tr}_R \hat{F}^2 &= A_R \text{tr}_f \hat{F}^2 \\ \text{tr}_R \hat{F}^4 &= B_R \text{tr}_f \hat{F}^4 + C_R (\text{tr}_f \hat{F}^2)^2. \end{aligned} \quad (3.5)$$

## 3.2 Circle compactification and Chern-Simons terms

Let us now compactify the theory on a circle and push it to the Coulomb branch. We use the conventions that six-dimensional fields have a hat, while the hat is dropped on all five-dimensional fields. At the massless level we obtain rank  $G$  five-dimensional  $U(1)$  gauge fields  $A^I$  from reducing  $\hat{A}$  and  $\dim G - \text{rank } G$  W-bosons  $A^\alpha$  labeled by the roots  $\alpha$  of  $G$ . In the Abelian sector we have  $n_{U(1)}$   $U(1)$  gauge fields  $A^m$  from the reduction of the  $\hat{A}^m$ . To the  $\dim G + n_{U(1)}$  gauge fields one finds associated Wilson line scalars  $\zeta^{\mathcal{I}}$ ,  $\zeta^m$ . The expansions of the six-dimensional gauge fields are analogous to (2.10). The scalars  $\zeta^{\mathcal{I}}$ ,  $\zeta^m$  parametrize the five-dimensional Coulomb branch

$$\langle \zeta^I \rangle \neq 0, \quad \langle \zeta^\alpha \rangle = 0, \quad \langle \zeta^m \rangle \neq 0. \quad (3.6)$$

These expectation values break the gauge group as  $G \times U(1)^{n_{U(1)}} \rightarrow U(1)^{\text{rank } G} \times U(1)^{n_{U(1)}}$  while rendering the W-bosons massive. The six-dimensional metric also contains the KK-vector  $A^0$ , the radius  $r$  and a five-dimensional metric  $g_{\mu\nu}$  in analogy to (2.9). For simplicity we assume that  $g_{\mu\nu}$  is Minkowskian. Furthermore, we find  $T_{\text{sd}} + T_{\text{asd}}$  Abelian vectors  $A^\alpha$  from reducing and dualizing  $\hat{B}_\alpha$ . The details on this reduction can be found, for example, in refs. [14, 15].

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<sup>8</sup>We omit Abelian-non-Abelian anomalies, since we will not account for these in the following discussion.



Considering the theory on the Coulomb branch, the charged fields and the W-bosons generically become massive. In addition, higher KK-modes admit a KK-mass contribution proportional to their level  $n$ . In summary, and in complete analogy to (2.13), the fields in the five-dimensional theory acquire masses

$$m = \begin{cases} m_{\text{CB}}^w + n m_{\text{KK}} = w_I \langle \zeta^I \rangle + \frac{n}{\langle r \rangle}, & \text{non-Abelian gauge group,} \\ m_{\text{CB}}^q + n m_{\text{KK}} = q_m \langle \zeta^m \rangle + \frac{n}{\langle r \rangle}, & \text{Abelian gauge group.} \end{cases} \quad (3.7)$$

In determining the effective theory for the massless modes one needs to systematically integrate out all massive states. As in three dimensions we focus on the Chern-Simons terms. These terms are topological in nature and can be corrected at one-loop level by all massive states. Five-dimensional Chern-Simons terms for  $U(1)$  gauge fields take the general form

$$S_{\text{CS}}^{\text{gauge}} = -\frac{1}{12} \int k_{\Lambda\Sigma\Theta} A^\Lambda \wedge F^\Sigma \wedge F^\Theta \quad (3.8a)$$

$$S_{\text{CS}}^{\text{grav}} = -\frac{1}{4} \int k_\Lambda A^\Lambda \wedge \text{tr}(\mathcal{R} \wedge \mathcal{R}), \quad (3.8b)$$

where  $k_{\Lambda\Sigma\Theta}$  and  $k_\Lambda$  are constants and  $\mathcal{R}$  is the five-dimensional curvature two-form. The collective index is  $\Lambda = (0, I, \alpha)$  or  $\Lambda = (0, m, \alpha)$ , respectively. The classical Chern-Simons coefficients in the circle reduced theory are found to be

$$\begin{aligned} k_{\alpha\beta\gamma} &= 0, & k_{0\alpha\beta} &= \Omega_{\alpha\beta}, & k_{I\alpha\beta} &= 0, \\ k_{m\alpha\beta} &= 0, & k_{00\alpha} &= 0, & k_{IJ\alpha} &= -\Omega_{\alpha\beta} b^\beta \mathcal{C}_{IJ}, \\ k_{mn\alpha} &= -\Omega_{\alpha\beta} b_{mn}^\beta, & k_{0I\alpha} &= 0, & k_{0m\alpha} &= 0, \\ k_\alpha &= -12 \Omega_{\alpha\beta} a^\beta. \end{aligned} \quad (3.9)$$

with  $\mathcal{C}_{IJ}$  the coroot intersection matrix defined in (C.3). Note that since we restrict either to pure non-Abelian or pure Abelian gauge groups, we do not display Chern-Simons coefficients involving both types of indices  $m, n$  and  $I, J$ .

In addition to the classical Chern-Simons terms (3.9) the effective theory can admit one-loop induced Chern-Simons terms from massive charged two-forms, spin- $1/2$  and spin- $3/2$  fermions [9–11]. Their contributions are given by

$$k_{\Lambda\Sigma\Theta}^{\text{loop}} = c_{AFF} q_\Lambda q_\Sigma q_\Theta \text{sign}(m), \quad (3.10)$$

$$k_\Lambda^{\text{loop}} = c_{ARR} q_\Lambda \text{sign}(m), \quad (3.11)$$

where the  $c_{AFF}, c_{ARR}$  are given in Table 3.1. The quantity  $q_\Lambda$  is the charge under  $A^\Lambda$  and  $\text{sign}(m)$  depends on the representation of the massive little group  $SO(4) \cong SU(2) \times SU(2)$  in the following way

$$\text{sign}(m) = \begin{cases} +1 & \text{for } (\frac{1}{2}, 0), (1, 0), (1, \frac{1}{2}), \\ -1 & \text{for } (0, \frac{1}{2}), (0, 1), (\frac{1}{2}, 1), \end{cases} \quad (3.12)$$

|           | spin- $1/2$ fermion | self-dual tensor | spin- $3/2$ fermion |
|-----------|---------------------|------------------|---------------------|
| $c_{AFF}$ | $1/2$               | $-2$             | $5/2$               |
| $c_{ARR}$ | $-1$                | $-8$             | $19$                |

Table 3.1: One-loop Chern-Simons term factors.

where we labeled representations of  $SU(2) \times SU(2)$  by their spins. In order to explicitly compute these terms we use the convention that the charge under the KK-vector is given as in (2.17). As in the preceding section we exploit zeta function regularization in order to treat the infinite KK-tower. Along the lines of [15] the relevant one-loop Chern-Simons coefficients of the circle reduced theory for the pure non-Abelian and pure Abelian theory respectively are evaluated to be

$$k_{IJK} = \frac{1}{2} \sum_R F_{1/2}(R) \sum_{w \in R} (2l_w + 1) w_I w_J w_K \text{sign}(m_{\text{CB}}^w), \quad (3.13a)$$

$$k_I = - \sum_R F_{1/2}(R) \sum_{w \in R} (2l_w + 1) w_I \text{sign}(m_{\text{CB}}^w), \quad (3.13b)$$

$$k_{mnp} = \frac{1}{2} \sum_q F_{1/2}(q) (2l_q + 1) q_m q_n q_p \text{sign}(m_{\text{CB}}^q), \quad (3.13c)$$

$$k_m = - \sum_q F_{1/2}(q) (2l_q + 1) q_m \text{sign}(m_{\text{CB}}^q), \quad (3.13d)$$

where  $l_w, l_q$  are defined in (2.19). The remaining one-loop Chern-Simons terms are listed for completeness in Appendix B. Note that the classical Chern-Simons coefficients do not receive any corrections. We proceed by establishing the precise correspondence of these one-loop Chern-Simons coefficients (3.13) and the anomaly conditions (3.4).

### 3.3 Non-Abelian models

We start by considering the purely non-Abelian simple gauge group  $G$  without Abelian factors and apply a basis transformation on the vectors in the circle reduced theory on the Coulomb branch. In analogy to the discussion of section 2, we first pick an arbitrary vector  $A^{\tilde{0}}$  out of the Cartan elements  $A^I$ . The remaining Cartan vectors are denoted by  $A^{\tilde{I}}$ , while the remaining  $\dim G - 1$  vectors of the  $A^{\mathcal{I}}$  are denoted by  $A^{\tilde{\mathcal{I}}}$ . The transformation then takes the form

$$\begin{pmatrix} \tilde{A}^0 \\ \tilde{A}^{\tilde{0}} \\ \tilde{A}^{\tilde{\mathcal{I}}} \\ \tilde{A}^\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \delta_{\tilde{\mathcal{I}}}^{\tilde{\mathcal{I}}} & 0 \\ \frac{1}{2}b^\alpha \mathcal{C}_{\tilde{0}\tilde{0}} & b^\alpha \mathcal{C}_{\tilde{0}\tilde{0}} & b^\alpha \mathcal{C}_{\tilde{0}\tilde{\mathcal{I}}} & \delta_\beta^\alpha \end{pmatrix} \cdot \begin{pmatrix} A^0 \\ A^{\tilde{0}} \\ A^{\tilde{\mathcal{I}}} \\ A^\beta \end{pmatrix}, \quad (3.14)$$

where  $\mathcal{C}_{IJ}$  is the coroot intersection matrix (C.3) of  $G$  and  $\mathcal{C}_{\tilde{I}\tilde{J}}$  vanishes for all non-Cartan directions. We denote quantities in the transformed basis by a tilde and again collect  $\tilde{A}^{\tilde{0}}, \tilde{A}^{\tilde{I}}$  in  $\tilde{A}^{\tilde{I}}$ . The group structure is simply  $\mathbb{Z}^{\text{rank } G}$ . One can check that the classical Chern-Simons terms (3.9) remain invariant under this transformation and the weights  $w$  of some representation  $R$  transform as in (2.21).

Let us next apply (3.14) to the one-loop Chern-Simons coefficients. Concretely, we first evaluate  $\tilde{k}_{IJK}$  by performing the loop-computation directly in the transformed basis making use of (3.13a) to find

$$\tilde{k}_{IJK} = \frac{1}{2} \sum_R F_{1/2}(R) \sum_{w \in R} (2\tilde{l}_w + 1) w_I w_J w_K \text{sign}(\tilde{m}_{\text{CB}}^w). \quad (3.15)$$

Then we can also transform  $\tilde{k}_{IJK}$  back to the old basis and perform the loop-computation there

$$\begin{aligned} \tilde{k}_{IJK} &= k_{IJK} - k_{IJ\alpha} b^\alpha \mathcal{C}_{\tilde{0}K} - k_{IK\alpha} b^\alpha \mathcal{C}_{\tilde{0}J} - k_{JK\alpha} b^\alpha \mathcal{C}_{\tilde{0}I} \\ &= \frac{1}{2} \sum_R F_{1/2}(R) \sum_{w \in R} (2l_w + 1) w_I w_J w_K \text{sign}(m_{\text{CB}}^w) \\ &\quad + b_\alpha b^\alpha \mathcal{C}_{IJ} \mathcal{C}_{\tilde{0}K} + b_\alpha b^\alpha \mathcal{C}_{IK} \mathcal{C}_{\tilde{0}J} + b_\alpha b^\alpha \mathcal{C}_{JK} \mathcal{C}_{\tilde{0}I}. \end{aligned} \quad (3.16)$$

Matching (3.15) with (3.16) we obtain using (2.25)

$$\sum_R F_{1/2}(R) \sum_{w \in R} w_I w_J w_K w_{\tilde{0}} = -b_\alpha b^\alpha \mathcal{C}_{IJ} \mathcal{C}_{\tilde{0}K} - b_\alpha b^\alpha \mathcal{C}_{IK} \mathcal{C}_{\tilde{0}J} - b_\alpha b^\alpha \mathcal{C}_{JK} \mathcal{C}_{\tilde{0}I}. \quad (3.17)$$

In Appendix C it is shown that this equation is equivalent to both pure non-Abelian gauge anomaly cancelation conditions (3.4c), (3.4d). More precisely, we show in Appendix C that it is possible to rewrite non-Abelian anomaly cancelation conditions in a convenient fashion. While in the usual presentation these conditions are written in terms of the factors  $A_R, B_R, C_R$ , which appear in the reduction of traces (3.5), we managed to reformulate them in terms of sums over weights, not involving any of the factors  $A_R, B_R, C_R$ . Applying this procedure to the remaining gauge one-loop Chern-Simons coefficients again yields the anomaly conditions (3.4c), (3.4d).

Interestingly we can also reproduce the mixed non-Abelian-gravitational anomaly (3.4a) by investigating the behavior of the gravitational Chern-Simons coefficient  $\tilde{k}_I$ . On the one hand, by directly evaluating the loop we obtain using (3.13b)

$$\tilde{k}_I = - \sum_R F_{1/2}(R) \sum_{w \in R} (2\tilde{l}_w + 1) w_I \text{sign}(\tilde{m}_{\text{CB}}^w). \quad (3.18)$$

On the other hand, in the old basis we find

$$\tilde{k}_I = k_I - b^\alpha \mathcal{C}_{\tilde{0}I} k_\alpha = - \sum_R F_{1/2}(R) \sum_{w \in R} (2l_w + 1) w_I \text{sign}(m_{\text{CB}}^w) + 12b^\alpha a_\alpha \mathcal{C}_{\tilde{0}I}. \quad (3.19)$$

Using (2.25) the matching of (3.18) and (3.19) yields the condition

$$\sum_R F_{1/2}(R) \sum_{w \in R} w_I w_{\bar{0}} = 6b^\alpha a_\alpha \mathcal{C}_{\bar{0}I}. \quad (3.20)$$

It is shown in [15] that the following identity holds true

$$\sum_{w \in R} w_I w_J = A_R \lambda(G) \mathcal{C}_{IJ} \quad \forall I, J. \quad (3.21)$$

Thus we can conclude that (3.20) coincides with the mixed non-Abelian-gravitational anomaly (3.4a). Applying these steps to  $k_0$  again yields the mixed anomaly.

### 3.4 Abelian models

Let us finally investigate a purely Abelian theory. In complete analogy we pick one arbitrary vector  $A^{\bar{0}}$  out of the  $A^m$  and denote the remaining ones by  $A^{\tilde{m}}$ . We define the transformation

$$\begin{pmatrix} \tilde{A}^0 \\ \tilde{A}^{\bar{0}} \\ \tilde{A}^{\tilde{m}} \\ \tilde{A}^\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & \delta_{\tilde{n}}^{\tilde{m}} & 0 \\ \frac{1}{2}b_{\bar{0}\bar{0}}^\alpha & b_{\bar{0}\bar{0}}^\alpha & b_{\bar{0}\tilde{n}}^\alpha & \delta_\beta^\alpha \end{pmatrix} \cdot \begin{pmatrix} A^0 \\ A^{\bar{0}} \\ A^{\tilde{n}} \\ A^\beta \end{pmatrix}, \quad (3.22)$$

which has precisely the same form as (2.27) and is therefore isomorphic to  $\mathbb{Z}^{n_{U(1)}}$ . The quantities in the transformed basis carry a tilde. However, in contrast to the four-dimensional setup, this transformation leaves all classical Chern-Simons terms (3.9) invariant and there is no need for an additional shift like (2.29).<sup>9</sup> Finally the charges transform as in (2.31).

We now show that the transformation of one-loop Chern-Simons terms under (3.22) yields the pure Abelian anomaly equations, as well as the mixed Abelian-gravitational anomaly. First consider  $\tilde{k}_{mnp}$  and evaluate the loop directly using (3.13c) as

$$\tilde{k}_{mnp} = \frac{1}{2} \sum_q F_{1/2}(q) (2\tilde{l}_q + 1) q_m q_n q_p \text{sign}(\tilde{m}_{\text{CB}}^q). \quad (3.23)$$

On the other hand we can transform  $\tilde{k}_{mnp}$  back to the old basis and do the loop there

$$\begin{aligned} \tilde{k}_{mnp} &= k_{mnp} - k_{mn\alpha} b_{\bar{0}p}^\alpha - k_{mp\alpha} b_{\bar{0}n}^\alpha - k_{np\alpha} b_{\bar{0}m}^\alpha \\ &= \frac{1}{2} \sum_q F_{1/2}(q) (2l_q + 1) q_m q_n q_p \text{sign}(m_{\text{CB}}^q) + (b_{mn}^\alpha b_{\bar{0}p}^\beta + b_{mp}^\alpha b_{\bar{0}n}^\beta + b_{np}^\alpha b_{\bar{0}m}^\beta) \Omega_{\alpha\beta}. \end{aligned} \quad (3.24)$$

---

<sup>9</sup>The reason for this can be understood via the M-/F-theory realization, since this time there is no  $G_4$ -flux that needs to be transformed.

Let us now match (3.23) and (3.24) and use the identity (2.34) to obtain the equation

$$\sum_q F_{1/2}(q) q_m q_n q_p q_{\tilde{0}} = -(b_{mn}^\alpha b_{0p}^\beta + b_{mp}^\alpha b_{0n}^\beta + b_{np}^\alpha b_{0m}^\beta) \Omega_{\alpha\beta}, \quad (3.25)$$

which is the pure Abelian anomaly condition (3.4e). Treating also the remaining gauge one-loop Chern-Simons coefficients in this way and making different choices for the vector  $A^{\tilde{0}}$  in the transformation (3.22) we obtain all pure Abelian anomaly conditions (with repetitions).

Lastly we can perform the loop-computation for  $\tilde{k}_m$  (3.13d)

$$\tilde{k}_m = - \sum_q F_{1/2}(q) (2\tilde{l}_q + 1) q_m \text{sign}(\tilde{m}_{\text{CB}}^q) \quad (3.26)$$

or transform it back again

$$\tilde{k}_m = k_m - b_{0m}^\alpha k_\alpha = - \sum_q F_{1/2}(q) (2l_q + 1) q_m \text{sign}(m_{\text{CB}}^q) + 12b_{0m}^\alpha a_\alpha. \quad (3.27)$$

Using (2.34) this results in the matching

$$\sum_q F_{1/2}(q) q_m q_{\tilde{0}} = 6b_{0m}^\alpha a^\beta \Omega_{\alpha\beta}, \quad (3.28)$$

which coincides with the mixed Abelian-gravitational anomaly equation (3.4b).

## 4 Anomalies on the circle in the M-/F-theory duality

In this section we provide the motivation for the transformations that led to the appearance of the higher-dimensional anomaly equations in the Kaluza-Klein theories on the circle. More precisely, we consider F-theory compactifications on elliptically fibered Calabi-Yau manifolds and investigate aspects of the effective action exploiting the duality to M-theory. After reviewing some properties of F-theory compactifications on elliptically fibered Calabi-Yau manifolds in subsection 4.1, we show that the transformations considered in section 2 and section 3 correspond to geometric symmetries.

The analysis for Abelian gauge symmetries is provided in subsection 4.2. We show that one is free to choose an arbitrary section of the elliptic fibration as the ‘*zero-section*’ and that different choices for zero-sections are related by transformations similar to (2.27), (3.22). Thus for Calabi-Yau fourfolds and threefolds the invariance of F-theory compactifications under zero-section changes proves the absence of Abelian anomalies in the effective action.

We transfer this observation to the non-Abelian case in subsection 4.3. More precisely, we start with transformations like (2.20), (3.14) and find retroactively their geometric manifestation. It turns out that these transformations map between different choices

of what we call the ‘*zero-node*’. When one resolves the singularities over a seven-brane divisor in the base the elliptic fiber splits into rank  $G + 1$  irreducible components. One arbitrary component is then to be chosen as the zero-node. However, we claim that F-theory compactifications on elliptically fibered Calabi-Yau manifolds are invariant under zero-node changes which then implies the absence of non-Abelian gauge anomalies.

## 4.1 F-theory on elliptically fibered Calabi-Yau manifolds

Let us review some basic facts about F-theory compactifications and their effective actions. F-theory compactified on elliptically fibered Calabi-Yau fourfolds yields  $\mathcal{N} = 1$  supergravity theories in four dimensions, while compactifications on elliptically fibered Calabi-Yau threefolds yield  $\mathcal{N} = (1, 0)$  supergravity theories in six dimensions. In both dimensions non-Abelian gauge groups can arise from singularities of the elliptic fibration, whereas Abelian gauge groups are induced if the fibration admits at least two sections.

In order to derive the effective action one can employ the duality between F-theory and M-theory. More concretely, one first reduces a general four- or six-dimensional supergravity theory on a circle, pushes the lower-dimensional theory onto the Coulomb branch, and matches it with the dual M-theory description. M-theory can be accessed via eleven-dimensional supergravity if one uses the same Calabi-Yau space, but with all singularities being resolved [26–31, 14, 15]. The derivation of the F-theory effective actions requires to match the supergravity data with geometric quantities using the three- or five-dimensional effective theories. Importantly, the naive matching procedure fails if one only restricts the considerations to the classical effective theory. In particular, the classical Chern-Simons terms in the circle reduced supergravity theory will not directly match with the M-theory side. In accord with the discussion of the previous sections one recalls that one-loop corrections to the Chern-Simons terms are induced by massive modes. These have to be taken into account to match the M-theory and F-theory reduction [12, 14, 13, 15–17]. Since the one-loop corrections to the Chern-Simons terms on the F-theory side are sensitive to the spectrum, the matching with the M-theory reduction reveals information about the spectrum in the F-theory effective theory in terms of geometric quantities of the resolved Calabi-Yau space and the background flux.

Let us further focus on Chern-Simons terms and study the matching procedure more precisely. First we have to introduce some geometric properties of the M-theory compactification. The resolved Calabi-Yau manifold is denoted by  $\hat{Y}$  and arises from an elliptic fibration over some base  $B$ . The projection to the base is denoted by  $\pi : \hat{Y} \rightarrow B$ . The linearly independent sections of the elliptic fibration are denoted by  $\sigma_0, \sigma_m$ , where we singled out one arbitrary section  $\sigma_0$  as the so-called zero-section. We assume that there is always at least one section. Next, there may be a divisor in the base  $B$  of the elliptic fibration over which the resolved singular fiber splits into a number of irreducible components which intersect as the affine Dynkin diagram of the gauge algebra. Fiberizing the latter over this divisor in  $B$  we obtain divisors of the whole fibration, which we denote by

$\Sigma_0, \Sigma_I$ . We again singled out an arbitrary component  $\Sigma_0$  as what we call the zero-node.<sup>10</sup>

We can now define a convenient basis of divisors  $D_\Lambda = (D_0, D_I, D_m, D_\alpha)$  in the elliptically fibered, resolved Calabi-Yau manifold  $\hat{Y}$  in the following way:

- The divisor  $D_0$  is obtained from  $\sigma_0$  supplemented by the shift (4.13) or the generalized shift (4.19), respectively. Expanding the M-theory three-form  $C_3$  along the corresponding two-form yields a vector that is identified with the KK-vector in the dual circle reduced F-theory setting.
- The exceptional divisors  $D_I$  are related to the fiber components  $\Sigma_I$  via (4.20). They correspond to the Cartan generators in the dual F-theory setting such that  $I = 1, \dots, \text{rank } G$ .
- The  $U(1)$  divisors  $D_m$  descend from the sections  $\sigma_m$  using the Shioda map (4.14). They correspond to  $U(1)$  gauge symmetries in the F-theory setting implying that  $m = 1, \dots, n_{U(1)}$ .
- The vertical divisors  $D_\alpha$  are obtained as  $\pi^{-1}(D_\alpha^b)$  from divisors  $D_\alpha^b$  of the base  $B$ . For each homologically independent divisor  $D_\alpha^b$  in  $B$  one finds an axion in four-dimensional F-theory compactifications and an (anti-) self-dual tensor in six dimensions, respectively.

For Calabi-Yau fourfolds we introduce vertical four-cycles  $\mathcal{C}^\alpha$  that are obtained as  $\pi^{-1}(\mathcal{C}_b^\alpha)$  from curves  $\mathcal{C}_b^\alpha$  in the base intersecting the  $D_\alpha^b$  as

$$\eta_\alpha{}^\beta = D_\alpha^b \cdot \mathcal{C}_b^\beta \quad (4.1)$$

with  $\eta_\alpha{}^\beta$  a full-rank matrix.

It turns out that the Chern-Simons coefficients (2.14), (3.8) in the M-theory compactification are given by the intersections<sup>11</sup> [29, 38, 39]

$$\Theta_{\Lambda\Sigma} = -\frac{1}{4} D_\Lambda \cdot D_\Sigma \cdot G_4 \quad \text{three dimensions,} \quad (4.2)$$

$$k_{\Lambda\Sigma\Theta} = D_\Lambda \cdot D_\Sigma \cdot D_\Theta \quad \text{five dimensions,} \quad (4.3)$$

$$k_\Lambda = D_\Lambda \cdot c_2 \quad \text{five dimensions,} \quad (4.4)$$

where  $G_4$  denotes the flux of the four-form field strength and  $c_2$  is the second Chern class of the Calabi-Yau threefold  $\hat{Y}$ . It is now possible to match these quantities with their counterparts in the circle reduced supergravity theory on the Coulomb branch. In the following we use the field theory notation as introduced in section 2 and section 3. The

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<sup>10</sup>This is reminiscent of the affine node in the F-theory literature, the node which is intersected by the zero-section. The zero-node may be considered as a generalization of this concept, since its choice is arbitrary and it is in particular not immediately related to the affine node of extended Dynkin diagrams.

<sup>11</sup>In the following Poincaré duality is always understood implicitly.

matching of intersection numbers with classical terms in the circle reduced theory gives for the fourfold [31, 40, 13]

$$\begin{aligned} D_I \cdot D_J \cdot \mathcal{C}^\alpha &= -\mathcal{C}_{IJ} b^\beta \eta_\beta^\alpha, & D_m \cdot D_n \cdot \mathcal{C}^\alpha &= -b_{mn}^\beta \eta_\beta^\alpha, \\ D_\alpha \cdot D_m \cdot G_4 &= -2\theta_{\alpha m}, \end{aligned} \quad (4.5)$$

where  $\theta_{\alpha m}$  are the axion gaugings defined in (2.5),  $b_{mn}^\alpha$ ,  $b^\alpha$  are the Green-Schwarz coefficients (2.6), and  $\eta_\beta^\alpha$  is the matrix defined in (4.1). For the threefold the classical Chern-Simons matching yields [14, 40, 15]

$$\begin{aligned} D_0 \cdot D_\alpha \cdot D_\beta &= \Omega_{\alpha\beta}, & D_\alpha \cdot c_2 &= -12a^\beta \Omega_{\alpha\beta}, \\ D_I \cdot D_J \cdot D_\alpha &= -\mathcal{C}_{IJ} b^\beta \Omega_{\alpha\beta}, & D_m \cdot D_n \cdot D_\alpha &= -b_{mn}^\beta \Omega_{\alpha\beta}, \end{aligned} \quad (4.6)$$

where  $a^\alpha$ ,  $b^\alpha$ ,  $b_{mn}^\alpha$  are the Green-Schwarz coefficients defined in (3.3).<sup>12</sup>

As already stressed, in the circle reduced theory one-loop corrections to the Chern-Simons terms have to be taken into account in order to perform a complete matching. For simplicity let us consider only the following matching of one-loop Chern-Simons terms in three dimensions

$$\begin{aligned} D_I \cdot D_J \cdot G_4 &= -4 \sum_R C(R) \sum_{w \in R} \left(l_w + \frac{1}{2}\right) w_I w_J \text{sign}(m_{\text{CB}}^w) \\ &\quad - 4 \sum_{\alpha \in \text{Adj}} \left(l_\alpha + \frac{1}{2}\right) \alpha_I \alpha_J \text{sign}(m_{\text{CB}}^\alpha), \end{aligned} \quad (4.7)$$

$$D_m \cdot D_n \cdot G_4 = -4 \sum_q C(q) \left(l_q + \frac{1}{2}\right) q_m q_n \text{sign}(m_{\text{CB}}^q), \quad (4.8)$$

and in five dimensions

$$\begin{aligned} D_I \cdot D_J \cdot D_K &= -\frac{1}{2} \sum_R H(R) \sum_{w \in R} (2l_w + 1) w_I w_J w_K \text{sign}(m_{\text{CB}}^w) \\ &\quad + \frac{1}{2} \sum_{\alpha \in \text{Adj}} (2l_\alpha + 1) \alpha_I \alpha_J \alpha_K \text{sign}(m_{\text{CB}}^\alpha), \end{aligned} \quad (4.9)$$

$$D_m \cdot D_n \cdot D_p = -\frac{1}{2} \sum_q H(q) (2l_q + 1) q_m q_n q_p \text{sign}(m_{\text{CB}}^q), \quad (4.10)$$

$$D_I \cdot c_2 = \sum_R H(R) \sum_{w \in R} (2l_w + 1) w_I \text{sign}(m_{\text{CB}}^w) + \sum_{\alpha \in \text{Adj}} (2l_\alpha + 1) \alpha_I \text{sign}(m_{\text{CB}}^\alpha), \quad (4.11)$$

$$D_m \cdot c_2 = \sum_q H(q) (2l_q + 1) q_m \text{sign}(m_{\text{CB}}^q). \quad (4.12)$$

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<sup>12</sup>Anomaly cancelation in six-dimensional F-theory compactifications has been studied in [41–45, 40].



Here  $C(R)/H(R)$  denote the number of chiral multiplets and hypermultiplets that transform in some representation  $R$  under the non-Abelian gauge group  $G$ . Accordingly,  $C(q)/H(q)$  counts the chiral multiplets and hypermultiplets with charge  $q$  under the  $U(1)$  gauge group factors. Of course, there are further equations appearing in the matching of one-loop terms with the intersection numbers. For simplicity we will not display all matchings in the following discussion, but rather include some additional comments on these later.

## 4.2 Abelian anomalies from zero-section changes

In this subsection we show how transformations similar to (2.27), (3.22) arise in the duality between M-theory and F-theory through different choices of the zero-section. For simplicity we consider only the pure Abelian case in the following. F-theory compactifications with both Abelian and non-Abelian groups have recently been studied intensively in [46–50, 15, 51–57]

As already announced, for the matching of the dual theories to work, one has to shift the zero-section  $\sigma_0$  in the definition of the base divisor. The correct way to do this is given by [58, 45, 13, 15]

$$D_0 = \sigma_0 - \frac{1}{2}(\sigma_0 \cdot \sigma_0 \cdot \mathcal{C}^\alpha) \eta^{-1}{}_\alpha{}^\beta D_\beta \quad \text{three dimensions,} \quad (4.13a)$$

$$D_0 = \sigma_0 - \frac{1}{2}(\sigma_0 \cdot \sigma_0 \cdot D^\alpha) D_\alpha \quad \text{five dimensions.} \quad (4.13b)$$

Recall that expanding the M-theory three-form along  $D_0$  gives the dual to the KK-vector. Furthermore, the remaining sections  $\sigma_m$  have to be shifted using the Shioda map in order to get the appropriate  $U(1)$  divisors [59, 60]. In absence of non-Abelian gauge symmetries the maps read

$$D_m = \sigma_m - D_0 - (\sigma_m \cdot D_0 \cdot \mathcal{C}^\alpha) \eta^{-1}{}_\alpha{}^\beta D_\beta \quad \text{three dimensions,} \quad (4.14a)$$

$$D_m = \sigma_m - D_0 - (\sigma_m \cdot D_0 \cdot D^\alpha) D_\alpha \quad \text{five dimensions.} \quad (4.14b)$$

It is now interesting to ask how these divisors transform if one chooses a different zero-section. Concretely fix one  $\sigma_{\tilde{0}}$  out of the  $\sigma_m$ . The remaining  $U(1)$  sections will be denoted by  $\sigma_{\tilde{m}}$ . Let us then take the original zero-section  $\sigma_0$  as an ordinary  $U(1)$  section and select  $\sigma_{\tilde{0}}$  as the new zero-section. The corresponding transformation of the divisors, dictated by (4.13), (4.14) is found to be

$$\begin{pmatrix} \tilde{D}_0 \\ \tilde{D}_{\tilde{0}} \\ \tilde{D}_{\tilde{m}} \\ \tilde{D}_\alpha \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & -\frac{1}{2}\mathcal{K}_{00}{}^\beta \\ 0 & -1 & 0 & \mathcal{K}_{00}{}^\beta \\ 0 & -1 & \delta_{\tilde{m}}^\alpha & \mathcal{K}_{00}{}^\beta - \mathcal{K}_{0\tilde{m}}{}^\beta \\ 0 & 0 & 0 & \delta_\alpha^\beta \end{pmatrix} \cdot \begin{pmatrix} D_0 \\ D_{\tilde{0}} \\ D_{\tilde{n}} \\ D_\beta \end{pmatrix}, \quad (4.15)$$

with the shorthand notation

$$\mathcal{K}_{\Lambda\Sigma}{}^\alpha := \eta^{-1}{}_\beta{}^\alpha D_\Lambda \cdot D_\Sigma \cdot \mathcal{C}^\beta \quad \text{three dimensions,} \quad (4.16a)$$

$$\mathcal{K}_{\Lambda\Sigma}{}^\alpha := D_\Lambda \cdot D_\Sigma \cdot D^\alpha \quad \text{five dimensions.} \quad (4.16b)$$

It is easy to check that this is indeed the transformation induced by zero-section changes using the intersection numbers listed in Appendix D. Inserting the quantities from the classical matchings (4.5), (4.6) for  $\mathcal{K}_{mn}{}^\alpha$  into (4.15) and taking the transpose and inverse gives the transformation of the vectors in the field theory. We point out that this map coincides with (2.27), (3.22) up to  $U(1)$  basis transformations in the higher-dimensional theories. Consequently the classical intersection numbers are only invariant up to  $U(1)$  basis transformations performed already in four and six dimensions.

We are now in the position to also identify the origin of the shifts (2.29) of the three-dimensional Chern-Simons terms. These were necessary in addition to the usual basis transformation when starting with four-dimensional pure Abelian models. In the M-theory manifestation we can see that the expression for the Chern-Simons coefficients  $\Theta_{\Lambda\Sigma}$  involves the  $G_4$ -flux (4.2). The choice of flux is constrained by several conditions [31, 61, 58, 13, 16], one of them stating  $\Theta_{0\alpha} \stackrel{!}{=} 0$ , which guarantees the absence of circle fluxes in the dual setting. In order to preserve this condition under the map (4.15) one also has to transform the flux according to

$$\tilde{G}_4 = G_4 - \eta^{-1}{}_\beta{}^\alpha (D_{\tilde{0}} \cdot D_\alpha \cdot G_4) \mathcal{C}^\beta. \quad (4.17)$$

We again stress that this transformation only appears for the Abelian models in four dimensions. It is now easy to work out the anomaly conditions by using the transformations of the one-loop Chern-Simons terms as done in the previous sections.

The fact that zero-section changes reproduce all Abelian anomalies has far-reaching consequences: It states that the invariance under the choice of the zero-section implies the cancelation of Abelian anomalies in the effective theory of F-theory compactifications on Calabi-Yau four- and threefolds. Hence this provides a proof for the absence of Abelian anomalies for F-theory compactifications on Calabi-Yau four- and threefolds considered here.

### 4.3 Non-Abelian anomalies from zero-node changes

In the following we describe the analog structure to the zero-section changes for non-Abelian gauge groups. We consider the non-Abelian version of the divisor transformation (4.15) and try to find a geometric pattern that reproduces these maps. It turns out that this can be achieved by introducing a generalized base divisor shift and a Shioda map for the Cartan divisors. These definitions depend on the choice of what we call the zero-node, and transformations of divisors are induced by changes of zero-nodes. We restrict to a setting with a simple non-Abelian gauge group  $G$  without Abelian factors and exactly one section.

The non-Abelian version of the transformation (4.15) is given by

$$\begin{pmatrix} \tilde{D}_0 \\ \tilde{D}_{\tilde{0}} \\ \tilde{D}_{\tilde{I}} \\ \tilde{D}_\alpha \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & -\frac{1}{2}\mathcal{K}_{\tilde{0}\tilde{0}}^\beta \\ 0 & -1 & 0 & \mathcal{K}_{\tilde{0}\tilde{0}}^\beta \\ 0 & -1 & \delta_{\tilde{I}}^{\tilde{J}} & \mathcal{K}_{\tilde{0}\tilde{0}}^\beta - \mathcal{K}_{\tilde{0}\tilde{I}}^\beta \\ 0 & 0 & 0 & \delta_\alpha^\beta \end{pmatrix} \cdot \begin{pmatrix} D_0 \\ D_{\tilde{0}} \\ D_{\tilde{J}} \\ D_\beta \end{pmatrix}, \quad (4.18)$$

where we split  $I = (\tilde{0}, \tilde{I})$  for some arbitrary choice of  $D_{\tilde{0}}$ . It is possible to find a geometric interpretation of this transformation. Over the codimension-one locus in the base where the seven-brane sits the fiber splits into rank  $G + 1$  irreducible components. Fiberizing these over the seven-brane divisor in the base we obtain rank  $G + 1$  divisors of the total space which we denote by  $\Sigma_0, \Sigma_I$ . The choice of  $\Sigma_0$  out of the rank  $G + 1$  divisors is arbitrary and we call it the zero-node. Having chosen a zero-node we can now write down a modified definition of the base divisor

$$D_0 = \sigma_0 - \frac{1}{2}(\sigma_0 \cdot \sigma_0 \cdot \mathcal{C}^\alpha) \eta^{-1}{}_\alpha{}^\beta D_\beta + (1 - \sigma_0 \cdot \Sigma_0 \cdot \mathcal{C}) \left[ \Sigma_0 - \frac{1}{2}(\Sigma_0 \cdot \Sigma_0 \cdot \mathcal{C}^\alpha) \eta^{-1}{}_\alpha{}^\beta D_\beta \right] \\ \text{three dimensions,} \quad (4.19a)$$

$$D_0 = \sigma_0 - \frac{1}{2}(\sigma_0 \cdot \sigma_0 \cdot D^\alpha) D_\alpha + (1 - \sigma_0 \cdot \Sigma_0 \cdot D) \left[ \Sigma_0 - \frac{1}{2}(\Sigma_0 \cdot \Sigma_0 \cdot D^\alpha) D_\alpha \right] \\ \text{five dimensions,} \quad (4.19b)$$

as well as a Shioda map for the Cartan divisors<sup>13</sup>

$$D_I = \Sigma_I + (1 - \sigma_0 \cdot \Sigma_0 \cdot \mathcal{C}) \left[ -\Sigma_0 + \left( \Sigma_0 \cdot (\Sigma_0 - \Sigma_I) \cdot \mathcal{C}^\alpha \right) \eta^{-1}{}_\alpha{}^\beta D_\beta \right] \\ + (\sigma_0 \cdot \Sigma_I \cdot \mathcal{C}) \left[ -\Sigma_I + (\Sigma_0 \cdot \Sigma_I \cdot \mathcal{C}^\alpha) \eta^{-1}{}_\alpha{}^\beta D_\beta \right] \quad \text{three dimensions,} \quad (4.20a)$$

$$D_I = \Sigma_I + (1 - \sigma_0 \cdot \Sigma_0 \cdot D) \left[ -\Sigma_0 + \left( \Sigma_0 \cdot (\Sigma_0 - \Sigma_I) \cdot D^\alpha \right) D_\alpha \right] \\ + (\sigma_0 \cdot \Sigma_I \cdot D) \left[ -\Sigma_I + (\Sigma_0 \cdot \Sigma_I \cdot D^\alpha) D_\alpha \right] \quad \text{five dimensions.} \quad (4.20b)$$

For a Calabi-Yau fourfold  $\hat{Y}$  we have introduced  $\mathcal{C}$  as the four-cycle  $\pi^{-1}(\mathcal{C}^b)$  in  $\hat{Y}$  obtained from a curve  $\mathcal{C}^b$  in  $B$  that intersects the seven-brane divisor exactly once. Similarly, for a Calabi-Yau threefold  $\hat{Y}$  we define  $D$  to be the divisor  $\pi^{-1}(D^b)$  in  $\hat{Y}$ , with  $D^b$  being a divisor in  $B$  intersecting the seven-brane divisor precisely once. The necessity to introduce  $\mathcal{C}$  and  $D$  is known already from the Abelian Shioda map in the presence of non-Abelian singularities, their precise construction can be looked up in [47]. It is important that the

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<sup>13</sup>We assume in the following that the zero-section intersects an affine node of the Dynkin diagram, which is certainly always true for toric constructions. We are confident that this generalizes to arbitrary geometries.

expressions  $(\sigma_0 \cdot \Sigma_0 \cdot \mathcal{C})$ ,  $(\sigma_0 \cdot \Sigma_I \cdot \mathcal{C})$ ,  $(\sigma_0 \cdot \Sigma_0 \cdot D)$ ,  $(\sigma_0 \cdot \Sigma_I \cdot D)$  equal to one if the respective node  $\Sigma_0$ ,  $\Sigma_I$  gets intersected by the zero-section  $\sigma_0$ , and are zero otherwise.

We note that in the case that the zero-section intersects the zero-node we obtain the usual F-theory definitions

$$D_0 = \sigma_0 - \frac{1}{2}(\sigma_0 \cdot \sigma_0 \cdot \mathcal{C}^\alpha) \eta^{-1}{}_\alpha{}^\beta D_\beta \quad \text{three dimensions,} \quad (4.21)$$

$$D_0 = \sigma_0 - \frac{1}{2}(\sigma_0 \cdot \sigma_0 \cdot D^\alpha) D_\alpha \quad \text{five dimensions,} \quad (4.22)$$

$$D_I = \Sigma_I. \quad (4.23)$$

One can check that under changes of zero-nodes the geometric definitions (4.19), (4.20) induce the transformations (4.18) on the divisors. We stress that the corresponding transformation to (4.18) of the vectors in the circle reduced theory on the Coulomb branch is again the same as (2.20), (3.14) up to basis transformations solely among the Cartan generators, not involving the KK-vector.<sup>14</sup> This means that anomalies are again canceled if (4.18) is a symmetry of the theory. Since independently of the chosen zero-node the geometric definitions (4.19), (4.20) always reproduce the appropriate intersection numbers that allow for a matching with the circle reduced theory on the Coulomb branch, we claim that the theory is invariant under zero-node changes. Invariance under zero-node changes then proves the absence of non-Abelian anomalies in F-theory compactifications on Calabi-Yau four- and threefolds.

We stress that for the completion of a rigorous proof of anomaly cancelation a deeper geometrical understanding of the definitions (4.19) and (4.20) would be desirable. Our key observation is that making these definitions and performing the subsequent zero-node changes actually correspond to geometric symmetries of the Calabi-Yau geometry by inspecting the intersection numbers. This symmetry has not been exploited so far and appears much less intuitive than the corresponding zero-section exchanges in the Abelian case.

At this stage it might be fruitful to go once again through the notions *zero-section*, *affine node*, *zero-node* for clarification:

- The choice of a particular *zero-section* corresponds to the choice of the corresponding Weierstrass model, more precisely the zero-section is mapped to the  $[z = 0]$ -section of the Weierstrass model.
- The *affine node* is precisely the node that is intersected by the zero-section. After the mapping to the (singular) Weierstrass model it is the only node which remains large, while all other nodes collapse to zero size. Importantly, by definition the affine node is linked to the choice of zero-section and cannot be picked independently.

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<sup>14</sup>Also in (2.20) we omitted the Green-Schwarz coefficients  $b^\alpha$  because they drop out in the calculations since  $\Theta_{\alpha I}$  vanishes.

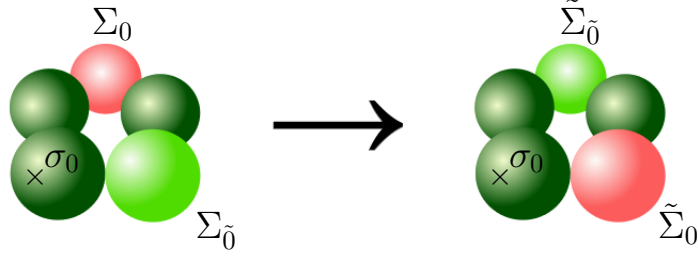


Figure 2: Two different choices for zero-nodes in a theory with  $A_4$  gauge algebra are depicted. The zero-nodes are colored in red, the remaining ones, corresponding to the Cartan generators, in green. Note that it is not necessary that the zero-node gets intersected by the zero-section.

- The *zero-node* is some kind of auxiliary bookkeeping device of the different possibilities for taking the F-theory limit. It can be any node of the associated affine Dynkin diagram including the ones that are not intersected by any section. The geometric meaning of this freedom is not clear yet and will be subject of future investigations.

For illustration we depict a generic change of zero-nodes for an  $A_4$ -model in Figure 2. It is however crucial to realize that in models other than  $A_n$  the zero-node does not have to be an affine node of the Dynkin diagram. This is possible since the choice of a zero-node is just an auxiliary step in the non-Abelian Shioda map. Indeed the affine node  $\Sigma_{\text{aff}}$  in the end always drops out in the definitions of the Cartan divisors. Let us illustrate this with an explicit calculation. Take a model with arbitrary gauge algebra and choose a zero-node  $\Sigma_0$  which cannot be interpreted as an affine node of the Dynkin diagram. We then first apply the Shioda map to all other nodes that are not intersected by the zero-section. We find e.g. in six dimensions

$$D_I = \Sigma_I - \Sigma_0 + \left( \Sigma_0 \cdot (\Sigma_0 - \Sigma_I) \cdot D^\alpha \right) D_\alpha. \quad (4.24)$$

Finally there is still the node left which is intersected by the zero-section, denoted by  $\Sigma_{\text{aff}}$ . Although, as already stated before this is an affine node of the Dynkin diagram, it however also defines a Cartan divisor via the Shioda map

$$D_{\text{aff}} = -\Sigma_0 + (\Sigma_0 \cdot \Sigma_0 \cdot D^\alpha) D_\alpha. \quad (4.25)$$

Note that the notation  $D_{\text{aff}}$  means just that this divisor is derived from the affine node, as a Cartan divisor it still belongs to a simple Lie-algebra. Importantly one realizes that the affine node  $\Sigma_{\text{aff}}$  finally drops out of all definitions, which is somehow expected.

## 5 Conclusions

In this paper we studied symmetries of four- and six-dimensional matter coupled gauge theories compactified on a circle. The considered transformations act on the circle-compactified theories by mixing the Kaluza-Klein vector arising when reducing the metric with the Kaluza-Klein zero-mode of a Cartan gauge field of a non-Abelian group or a  $U(1)$  Abelian gauge field. They are induced by a large gauge transformation in the higher-dimensional theory with support in the circle direction. Under this action the whole Kaluza-Klein tower of fields gets rearranged. If the higher-dimensional theory does not cancel anomalies, this has profound implications for the lower-dimensional one-loop effective theory for the massless modes. The considered transformations allow us to exactly extract the anomaly conditions from the effective theory.

Our focus was on the Chern-Simons terms in the three- and five-dimensional effective theory that are known to receive one-loop corrections when integrating out massive fields, which are precisely the Kaluza-Klein modes of higher-dimensional fields that contribute to the anomaly. Their mass depends on the Coulomb branch parameters and the circle radius if they are excited Kaluza-Klein modes. Depending on the Coulomb branch parameters and the circle radius there seemingly exists a discrete set of infinitely many effective theories for the massless modes differing by their one-loop Chern-Simons terms. We have shown that precisely when anomalies are canceled the transformations mixing gauge fields and the Kaluza-Klein vector identify identical effective theories. This is in accord with the fact that large gauge transformations are only symmetries of the one-loop quantum theory if anomalies are canceled. In other words, the naive moduli space of the Coulomb branch parameters in an anomaly free theory can be modded out by the shifts (2.22) to label inequivalent theories. Note that although it was already known that Chern-Simons terms know about higher-dimensional anomalies, the direct relation to the anomaly conditions and the periodicity of the Coulomb branch parameters seems to have not appeared in the literature before.

A closer look on the considered symmetry transformations shows that they act in a rather non-trivial way in the Kaluza-Klein perspective. For example, for a non-Abelian gauge group already within a single representation of matter fields the shifts of the Kaluza-Klein level act differently on the fields depending on the weight of the field in the representation. Furthermore, this is accompanied with the fact that in non-Abelian gauge theories the gauge transformation of the transformed Kaluza-Klein vector is not anymore that of a  $U(1)$  field but it rather mixes with the non-Abelian gauge transformations. All these modification have to be taken into account when computing the one-loop effective theory. In addition, we have found that in four-dimensional theories with  $U(1)$  gauge factors and Green-Schwarz axions an additional shift of the Chern-Simons terms has to be performed. This shift ensures the absence of circle fluxes for the axions that otherwise would complicate the one-loop computation. Nevertheless, it would be interesting to evaluate the effective theory for this more general situation directly.

The original motivation for the study of the symmetries came from the M-/F-theory

duality. More precisely F-theory on an elliptically fibered Calabi-Yau manifold compactified on an additional circle and pushed to the Coulomb branch is dual to M-theory on the resolved Calabi-Yau space. The effective action of the F-theory compactification is then obtained from matching a circle reduced supergravity theory with the M-theory compactification. This implies that a study of anomaly cancelation for F-theory effective actions requires to approach them using a lower-dimensional perspective. Up to this work, anomaly cancelation was mostly studied on a case-by-case basis, by extracting the spectrum and Green-Schwarz terms for a given geometry and background flux. All studied examples were shown to be anomaly free. Only when restricting to special geometries and a subset of anomalies it was possible to show anomaly cancelation generally. In this work we proposed a general argument that geometric symmetries actually ensure cancelation of anomalies for a general class of Calabi-Yau manifolds.

In F-theory compactifications with only Abelian gauge symmetries there is a clear identification of the transformation mixing the Kaluza-Klein vector and the  $U(1)$  gauge fields. Recall that the Kaluza-Klein vector corresponds to the zero-section of the elliptic fibration while further  $U(1)$  gauge fields correspond to additional sections of the geometry. However, since the M-theory to F-theory duality does not depend on the choice of the zero-section one is free to pick any section of the fibration. It turned out that the different choices of zero-sections exactly induce the transformations encountered in the study of anomalies. Since this is actually a geometrical symmetry this shows the general cancelation of pure Abelian anomalies in F-theory compactifications on Calabi-Yau manifolds. Note that we found that changing the choice of the zero-section implies that the four-form flux  $G_4$  has to be shifted in order to avoid the appearance of circle fluxes. Generalizing the one-loop computation to include circle fluxes we believe that the same results can be obtained without shifting  $G_4$ . This implies that more general classes of  $G_4$  might be allowed in F-theory.

For non-Abelian gauge theories in F-theory the identification of an analog geometric symmetry turned out to be more involved. Nevertheless, we were able to propose a geometric symmetry, corresponding to zero-node changes, following our field theory insights. Over a seven-brane divisor in the fully resolved geometry the fiber splits into irreducible components intersecting as the affine Dynkin diagram of the gauge algebra. One of these components is chosen to be the zero-node. Using all components and the picked zero-node we introduced a refined definition of divisors corresponding to the Cartan elements of the gauge group and the Kaluza-Klein vector. These were shown to leave the classical intersection numbers invariant up to a higher-dimensional basis transformation. We identified our symmetry transformation as the freedom to pick a zero-node. This shows the general absence of non-Abelian gauge anomalies in F-theory compactifications on Calabi-Yau manifolds. It would be interesting to gain a better geometric understanding for the freedom of picking a zero-node and the related refined constructions of divisors. Furthermore, it would also be desirable to generalize the analysis of phases of lower-dimensional gauge theories of [12, 33–37] to fully include the Kaluza-Klein vector.

Let us comment on possible generalizations of our findings. It is an exciting question

whether our approach generalizes to more complicated compactifications of consistent supergravity theories or string theory. Moreover, it would be interesting to study other anomalies in a similar spirit. We are confident that our perspective straightforwardly generalizes to the treatment of mixed Abelian-non-Abelian anomalies. More involved are purely gravitational anomalies, in which case one would expect that the mixing of the Kaluza-Klein vector with the spin connection needs to be studied. Finally, it would be very interesting to address anomalies for other symmetries, such as conformal anomalies or R-symmetry anomalies.

## Acknowledgments

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## A Identities on the Coulomb branch

In this section we show the central identities

$$q_{\tilde{0}} = \left(l_q + \frac{1}{2}\right) \text{sign}(m_{\text{CB}}^q) - \left(\tilde{l}_q + \frac{1}{2}\right) \text{sign}(\tilde{m}_{\text{CB}}^q), \quad (\text{A.1})$$

$$w_{\tilde{0}} = \left(l_w + \frac{1}{2}\right) \text{sign}(m_{\text{CB}}^w) - \left(\tilde{l}_w + \frac{1}{2}\right) \text{sign}(\tilde{m}_{\text{CB}}^w). \quad (\text{A.2})$$

under the transformations (2.20), (2.27), (3.14), (3.22). We only prove the first identity, since the second one works out in exactly the same way. Consider a massive mode with charge vector  $q$  under Abelian gauge bosons  $A^m$ , Coulomb branch mass  $m_{\text{CB}}^q$  and KK-level  $n$ . We pick one  $A^{\tilde{0}}$  and perform one of the basis changes (2.27), (3.22). It is important to notice that these transformations leave all VEVs invariant except of

$$\langle \zeta^{\tilde{0}} \rangle \mapsto \langle \zeta^{\tilde{0}} \rangle - \frac{1}{\langle r \rangle}. \quad (\text{A.3})$$

One can then easily show that the sign function fulfills

$$\text{sign}(m_{\text{CB}}^q + n m_{\text{KK}}) = \text{sign}(\tilde{m}_{\text{CB}}^q + (n + q_{\tilde{0}}) m_{\text{KK}}). \quad (\text{A.4})$$

Depending on the sign of the Coulomb branch masses we have to investigate four different cases:

1.  $\text{sign}(m_{\text{CB}}^q) > 0$

The integer quantity  $l_q$  is then defined via the following property

$$\text{sign}(m_{\text{CB}}^q - l_q m_{\text{KK}}) > 0 \quad \wedge \quad \text{sign}(m_{\text{CB}}^q - (l_q + 1) m_{\text{KK}}) < 0. \quad (\text{A.5})$$



Using (A.4) we find

$$\text{sign}(\tilde{m}_{\text{CB}}^q - (l_q - q_{\bar{0}}) m_{\text{KK}}) > 0 \quad \wedge \quad \text{sign}(\tilde{m}_{\text{CB}}^q - (l_q - q_{\bar{0}} + 1) m_{\text{KK}}) < 0. \quad (\text{A.6})$$

Depending on the sign of  $m_{\text{CB}}^q$  we can now read off  $\tilde{l}_q$

$$\tilde{l}_q = l_q - q_{\bar{0}} \quad \text{for } \text{sign}(\tilde{m}_{\text{CB}}^q) > 0 \quad (\text{A.7a})$$

$$\tilde{l}_q = -l_q + q_{\bar{0}} - 1 \quad \text{for } \text{sign}(\tilde{m}_{\text{CB}}^q) < 0. \quad (\text{A.7b})$$

## 2. $\text{sign}(m_{\text{CB}}^q) < 0$

Now  $l_q$  is defined as

$$\text{sign}(m_{\text{CB}}^q + (l_q + 1) m_{\text{KK}}) > 0 \quad \wedge \quad \text{sign}(m_{\text{CB}}^q + l_q m_{\text{KK}}) < 0. \quad (\text{A.8})$$

With (A.4) we get

$$\text{sign}(\tilde{m}_{\text{CB}}^q + (l_q + q_{\bar{0}} + 1) m_{\text{KK}}) > 0 \quad \wedge \quad \text{sign}(\tilde{m}_{\text{CB}}^q + (l_q + q_{\bar{0}}) m_{\text{KK}}) < 0. \quad (\text{A.9})$$

From this we can again determine  $\tilde{l}_q$

$$\tilde{l}_q = -l_q - q_{\bar{0}} - 1 \quad \text{for } \text{sign}(\tilde{m}_{\text{CB}}^q) > 0 \quad (\text{A.10a})$$

$$\tilde{l}_q = l_q + q_{\bar{0}} \quad \text{for } \text{sign}(\tilde{m}_{\text{CB}}^q) < 0. \quad (\text{A.10b})$$

It is now easy to check that the relations (A.7), (A.10) are summarized as

$$q_{\bar{0}} = \left(l_q + \frac{1}{2}\right) \text{sign}(m_{\text{CB}}^q) - \left(\tilde{l}_q + \frac{1}{2}\right) \text{sign}(\tilde{m}_{\text{CB}}^q). \quad (\text{A.11})$$

In complete analogy one proves the identity

$$w_{\bar{0}} = \left(l_w + \frac{1}{2}\right) \text{sign}(m_{\text{CB}}^w) - \left(\tilde{l}_w + \frac{1}{2}\right) \text{sign}(\tilde{m}_{\text{CB}}^w). \quad (\text{A.12})$$

Finally considering the basis transformations in the M-/F-theory setting (4.15), (4.18) one can show in the same way

$$q_{\bar{0}} = -\left(l_q + \frac{1}{2}\right) \text{sign}(m_{\text{CB}}^q) + \left(\tilde{l}_q + \frac{1}{2}\right) \text{sign}(\tilde{m}_{\text{CB}}^q), \quad (\text{A.13})$$

$$w_{\bar{0}} = -\left(l_w + \frac{1}{2}\right) \text{sign}(m_{\text{CB}}^w) + \left(\tilde{l}_w + \frac{1}{2}\right) \text{sign}(\tilde{m}_{\text{CB}}^w). \quad (\text{A.14})$$

Note the sign change compared to the relations before.

## B One-loop calculations

In this section we perform the loop-calculations to find the corrections to the Chern-Simons terms. Around (2.16) it is noted that in three dimensions one-loop Chern-Simons terms arise from spin- $1/2$  fermions, the contribution of a single Dirac field given by [6–8]

$$\Theta_{\Lambda\Sigma}^{\text{loop}} = \frac{1}{2} q_{\Lambda} q_{\Sigma} \text{sign}(m). \quad (\text{B.1})$$

In five dimensions one-loop Chern-Simons terms are induced by spin- $1/2$  and spin- $3/2$  fermions as well as two-forms, as already stated around (3.10), (3.11). The corrections originating from a single field are [9–11]

$$k_{\Lambda\Sigma\Theta}^{\text{loop}} = c_{AFF} q_{\Lambda} q_{\Sigma} q_{\Theta} \text{sign}(m), \quad (\text{B.2})$$

$$k_{\Lambda}^{\text{loop}} = c_{AR\mathcal{R}} q_{\Lambda} \text{sign}(m). \quad (\text{B.3})$$

The quantities  $c_{AFF}$ ,  $c_{AR\mathcal{R}}$  are listed in Table 3.1. Since we are treating circle compactified theories in this paper, the full contributions to the one-loop Chern-Simons terms are generically infinite sums over KK-modes, which need to be treated with zeta function regularization. In these calculations four different types of sums do appear in general:

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \text{sign}(x+n), & \quad \sum_{n=-\infty}^{+\infty} n \text{sign}(x+n), \\ \sum_{n=-\infty}^{+\infty} n^2 \text{sign}(x+n), & \quad \sum_{n=-\infty}^{+\infty} n^3 \text{sign}(x+n). \end{aligned} \quad (\text{B.4})$$

Using zeta function regularization

$$\sum_{n=1}^{\infty} n \mapsto \zeta(-1) = -\frac{1}{12}, \quad \sum_{n=1}^{\infty} n^3 \mapsto \zeta(-3) = \frac{1}{120}. \quad (\text{B.5})$$

these sums become

$$\sum_{n=-\infty}^{+\infty} \text{sign}(x+n) = (2l+1) \text{sign}(x), \quad (\text{B.6})$$

$$\sum_{n=-\infty}^{+\infty} n \text{sign}(x+n) = -\frac{1}{6} - l(l+1), \quad (\text{B.7})$$

$$\sum_{n=-\infty}^{+\infty} n^2 \text{sign}(x+n) = \frac{1}{3} l(l+1)(2l+1) \text{sign}(x), \quad (\text{B.8})$$

$$\sum_{n=-\infty}^{+\infty} n^3 \text{sign}(x+n) = \frac{1}{60} - \frac{1}{2} l^2(l+1)^2, \quad (\text{B.9})$$

where

$$l := \lfloor |x| \rfloor, \quad (\text{B.10})$$

making use of the floor function  $\lfloor \cdot \rfloor$ .

In order to find the full set of one-loop Chern-Simons coefficients for four-dimensional theories on the circle as introduced in section 2 we note that four-dimensional Weyl fermions reduce to three-dimensional Dirac fermions with  $\text{sign}(m) = \text{sign}(m_{\text{CB}} + nm_{\text{KK}})$  or  $\text{sign}(m) = -\text{sign}(m_{\text{CB}} + nm_{\text{KK}})$  for a former left-handed or right-handed spinor, respectively. We evaluate for the pure non-Abelian theory [15, 16]

$$\Theta_{00} = \frac{1}{3} \sum_R F(R) \sum_{w \in R} l_w (l_w + 1) \left(l_w + \frac{1}{2}\right) \text{sign}(m_{\text{CB}}^w), \quad (\text{B.11a})$$

$$\Theta_{0I} = \frac{1}{12} \sum_R F(R) \sum_{w \in R} (1 + 6 l_w (l_w + 1)) w_I, \quad (\text{B.11b})$$

$$\Theta_{IJ} = \sum_R F(R) \sum_{w \in R} \left(l_w + \frac{1}{2}\right) w_I w_J \text{sign}(m_{\text{CB}}^w), \quad (\text{B.11c})$$

where the sums are over all representations and all weights of a given representation. In the pure Abelian theory we obtain

$$\Theta_{00} = \frac{1}{3} \sum_q F(q) l_q (l_q + 1) \left(l_q + \frac{1}{2}\right) \text{sign}(m_{\text{CB}}^q), \quad (\text{B.12a})$$

$$\Theta_{0m} = \frac{1}{12} \sum_q F(q) (1 + 6 l_q (l_q + 1)) q_m, \quad (\text{B.12b})$$

$$\Theta_{mn} = \sum_q F(q) \left(l_q + \frac{1}{2}\right) q_m q_n \text{sign}(m_{\text{CB}}^q). \quad (\text{B.12c})$$

In six-dimensional theories on the circle following the pattern of section 3 we realize that Weyl spinors reduce to Dirac spinors and the KK-modes of former (anti-)self-dual tensors are massive two-forms with first order kinetic terms, the corresponding Lagrangians can be looked up e.g. in [11, 62, 63]. The contributions of these fields to the loop-corrections can then be inferred from Table 3.1. We note that  $\text{sign}(m)$  on these modes reads

$$\text{sign}(m) = \begin{cases} +\text{sign}(m_{\text{CB}} + n m_{\text{KK}}) & \text{for } (\frac{1}{2}, 0), (1, 0), (1, \frac{1}{2}), \\ -\text{sign}(m_{\text{CB}} + n m_{\text{KK}}) & \text{for } (0, \frac{1}{2}), (0, 1), (\frac{1}{2}, 1), \end{cases} \quad (\text{B.13})$$

where we labeled the representations of the former six-dimensional massless fields under the massless little group in six dimensions  $SU(2) \times SU(2)$  by their spins. Furthermore note that because of the (anti-)self-duality condition of the tensors in six dimensions

the contribution of a corresponding KK-mode is only half the one listed in Table 3.1.<sup>15</sup> Finally the corrections for the pure non-Abelian theory are given by [15]

$$k_{000} = \frac{1}{120} \left( 2(T_{sd} - T_{asd}) - F_{1/2} - 5F_{3/2} \right) + \frac{1}{4} \sum_R F_{1/2}(R) \sum_{w \in R} l_w^2 (l_w + 1)^2, \quad (\text{B.14a})$$

$$k_{00I} = \frac{1}{6} \sum_R F_{1/2}(R) \sum_{w \in R} l_w (l_w + 1) (2l_w + 1) w_I \text{sign}(m_{\text{CB}}^w), \quad (\text{B.14b})$$

$$k_{0IJ} = \frac{1}{12} \sum_R F_{1/2}(R) \sum_{w \in R} (1 + 6 l_w (l_w + 1)) w_I w_J, \quad (\text{B.14c})$$

$$k_{IJK} = \frac{1}{2} \sum_R F_{1/2}(R) \sum_{w \in R} (2l_w + 1) w_I w_J w_K \text{sign}(m_{\text{CB}}^w), \quad (\text{B.14d})$$

$$k_0 = \frac{1}{6} \left( 19F_{3/2} - F_{1/2} - 4(T_{sd} - T_{asd}) \right) - \sum_R F_{1/2}(R) \sum_{w \in R} l_w (l_w + 1), \quad (\text{B.14e})$$

$$k_I = - \sum_R F_{1/2}(R) \sum_{w \in R} (2l_w + 1) w_I \text{sign}(m_{\text{CB}}^w), \quad (\text{B.14f})$$

and for the Abelian theory they read

$$k_{000} = \frac{1}{120} \left( 2(T_{sd} - T_{asd}) - F_{1/2} - 5F_{3/2} \right) + \frac{1}{4} \sum_q F_{1/2}(q) l_q^2 (l_q + 1)^2, \quad (\text{B.15a})$$

$$k_{00m} = \frac{1}{6} \sum_q F_{1/2}(q) l_q (l_q + 1) (2l_q + 1) q_m \text{sign}(m_{\text{CB}}^q), \quad (\text{B.15b})$$

$$k_{0mn} = \frac{1}{12} \sum_q F_{1/2}(q) (1 + 6 l_q (l_q + 1)) q_m q_n, \quad (\text{B.15c})$$

$$k_{mnp} = \frac{1}{2} \sum_q F_{1/2}(q) (2l_q + 1) q_m q_n q_p \text{sign}(m_{\text{CB}}^q), \quad (\text{B.15d})$$

$$k_0 = \frac{1}{6} \left( 19F_{3/2} - F_{1/2} - 4(T_{sd} - T_{asd}) \right) - \sum_q F_{1/2}(q) l_q (l_q + 1), \quad (\text{B.15e})$$

$$k_m = - \sum_q F_{1/2}(q) (2l_q + 1) q_m \text{sign}(m_{\text{CB}}^q). \quad (\text{B.15f})$$

---

<sup>15</sup>This is also the case if one reduces Majorana-Weyl rather than Weyl spinors on a circle to five dimensions.

## C Lie theory conventions and trace identities

In this section we summarize our conventions for the Lie algebra theory in this paper. Furthermore we show that the factors appearing in trace reductions of some representation of a simple Lie algebra can be related to the sum over the weights in that representation. This will allow us to relate one-loop Chern-Simons terms to non-Abelian anomaly cancelation conditions since in the former sums over weights are evaluated, while in the latter factors of trace reductions appear.

Consider a simple Lie algebra  $\mathfrak{g}$ . We define a (preliminary) basis of Cartan generators  $\{\tilde{T}_i\}$  enforcing

$$\mathrm{tr}_f(\tilde{T}_i \tilde{T}_j) = \delta_{ij}, \quad (\text{C.1})$$

which encodes the normalization of the root lattice, and the trace  $\mathrm{tr}_f$  is taken in the fundamental representation. We denote the simple roots by  $\alpha_I$ ,  $I = 1, \dots, \mathrm{rank} \mathfrak{g}$ , the simple coroots are denoted by  $\alpha_I^\vee := \frac{2\alpha_I}{\langle \alpha_I, \alpha_I \rangle}$ . It turns out that for the considerations in this paper the coroot-basis  $\{T_I\}$  for the Cartan-subalgebra is more convenient, it is given by

$$T_I := \frac{2\alpha_I^\vee \tilde{T}_i}{\langle \alpha_I, \alpha_I \rangle} \quad (\text{C.2})$$

with  $\alpha_I^i$  the components of the simple roots. We furthermore define the (normalized) coroot intersection matrix  $\mathcal{C}_{IJ}$  as

$$\mathcal{C}_{IJ} = \frac{1}{\lambda(\mathfrak{g})} \langle \alpha_I^\vee, \alpha_J^\vee \rangle, \quad (\text{C.3})$$

with

$$\lambda(\mathfrak{g}) = \frac{2}{\langle \alpha_{\max}, \alpha_{\max} \rangle}, \quad (\text{C.4})$$

where  $\alpha_{\max}$  is the root of maximal length. The normalization of the Cartan generators  $T_I$  (in the coroot basis) can then be checked to take the form

$$\mathrm{tr}_f(T_I T_J) = \lambda(\mathfrak{g}) \mathcal{C}_{IJ}. \quad (\text{C.5})$$

Furthermore for some weight  $w$  the Dynkin labels are defined as

$$w_I := \langle \alpha_I^\vee, w \rangle. \quad (\text{C.6})$$

Finally in Table C.1 we display the numbering of the nodes in the Dynkin diagrams, the definition of the fundamental representations of all simple Lie algebras, as well as the values for the normalization factors  $\lambda(\mathfrak{g})$  in our conventions.

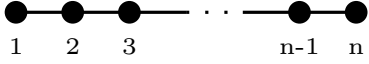
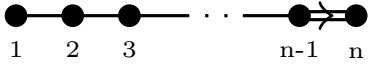
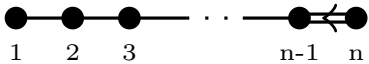
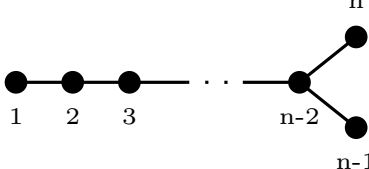
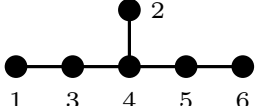
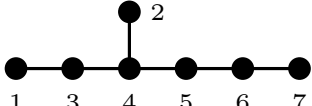
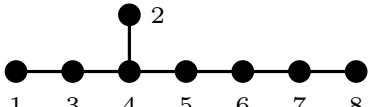
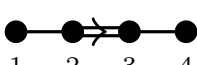

| algebra | Dynkin diagram  | fundamental representation  | $\lambda(\mathfrak{g})$ |
|---------|---|-----------------------------|-------------------------|
| $A_n$   |    | $(1, 0, 0, \dots, 0, 0)$    | 1                       |
| $B_n$   |    | $(1, 0, 0, \dots, 0, 0)$    | 2                       |
| $C_n$   |    | $(1, 0, 0, \dots, 0, 0)$    | 1                       |
| $D_n$   |    | $(1, 0, 0, \dots, 0, 0, 0)$ | 2                       |
| $E_6$   |   | $(0, 0, 0, 0, 0, 1)$        | 6                       |
| $E_7$   |  | $(0, 0, 0, 0, 0, 0, 1)$     | 12                      |
| $E_8$   |  | $(0, 0, 0, 0, 0, 0, 0, 1)$  | 60                      |
| $F_4$   |  | $(0, 0, 0, 1)$              | 6                       |
| $G_2$   |  | $(1, 0)$                    | 2                       |

Table C.1: Conventions for the simple Lie algebras.

## C.1 Cubic trace identities

We now show that the conditions

$$\sum_R F(R) \sum_{w \in R} w_I w_J w_K = 0 \quad \forall I, J, K, \quad (\text{C.7})$$

which appear in (2.26), are, depending on the choice of indices, either trivially fulfilled or equivalent to the cancelation of pure non-Abelian anomalies (2.7a)

$$\sum_R F(R) V_R = 0, \quad (\text{C.8})$$

where  $V_R$  appears in the trace reduction

$$\text{tr}_R \hat{F}^3 = V_R \text{tr}_f \hat{F}^3. \quad (\text{C.9})$$

Expanding the traces we can write (C.9) as

$$F^I F^J F^K \sum_{w \in R} w_I w_J w_K = F^I F^J F^K V_R \sum_{w^f} w_I^f w_J^f w_K^f, \quad (\text{C.10})$$

where  $F = F^I T_I$  and we sum over all weights, in particular  $w^f$  denote the weights of the fundamental representation. Considering this equation as a generating function we find

$$\sum_{w \in R} w_I w_J w_K = V_R \sum_{w^f} w_I^f w_J^f w_K^f. \quad (\text{C.11})$$

The key point is now to try to generally evaluate the sum over the fundamental weights on the right hand side. This procedure indeed will allow us to relate the factor  $V_R$  to a certain sum over the weights in the representation  $R$ , which appears in the calculation of one-loop Chern-Simons terms. In the following we carry this out for all simple Lie algebras.

### 1. $\mathbf{A}_1, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n, \mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8, \mathbf{F}_4, \mathbf{G}_2$

For these algebras there exists no cubic Casimir operator which is why non-Abelian anomalies are always trivially absent and one therefore defines  $V_R = 0$ . Via (C.11) the condition from the one-loop Chern-Simons matching (C.7) is then equivalent to the non-Abelian anomaly cancelation condition (C.8).<sup>16</sup>

### 2. $\mathbf{A}_{n \neq 1}$

For  $A_{n \neq 1}$  there exists a cubic Casimir and we start by explicitly evaluating the traces over the fundamental weights for different index choices.

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<sup>16</sup>Note also that the condition  $\sum_{w^f} w_I^f w_J^f w_K^f = 0 \quad \forall I, J, K$  precisely means that there is no cubic Casimir and one then also has by definition  $V_R = 0$ .

- (a)  $I = J = K$   
We calculate

$$\sum_{w^f} (w_I^f)^3 = 0 \quad (\text{C.12})$$

such that we can conclude using (C.11)

$$\sum_{w \in R} (w_I)^3 = 0. \quad (\text{C.13})$$

The corresponding Chern-Simons matching (C.7) is therefore trivial and imposes no restrictions on the spectrum.

- (b)  $I = K \neq J$   
Now we evaluate

$$\sum_{w^f} (w_I^f)^2 w_J^f = (I - J) \mathcal{C}_{IJ}. \quad (\text{C.14})$$

With (C.11) the Chern-Simons matching (C.7) in this case becomes

$$\sum_R F(R) V_R (I - J) \mathcal{C}_{IJ} = 0, \quad (\text{C.15})$$

which is equivalent to the anomaly condition (C.8).

- (c)  $I \neq J \neq K$   
Finally it turns out that

$$\sum_{w^f} w_I^f w_J^f w_K^f = 0, \quad (\text{C.16})$$

which is why the Chern-Simons matching is again trivial like in the case  $I = J = K$ .

To put it in a nutshell we have shown that the Chern-Simons matching (C.7) is completely equivalent to the anomaly cancelation conditions (C.8) for all simple Lie algebras.

## C.2 Quartic trace identities

Let us perform the same steps as in the last subsection now for quartic traces. More precisely we show that the matching of Chern-Simons coefficients (3.17)

$$\sum_R F_{1/2}(R) \sum_{w \in R} w_I w_J w_K w_L = -b^\alpha b^\beta \Omega_{\alpha\beta} (\mathcal{C}_{IJ} \mathcal{C}_{KL} + \mathcal{C}_{IK} \mathcal{C}_{JL} + \mathcal{C}_{IL} \mathcal{C}_{JK}) \quad (\text{C.17})$$



is equivalent to the six-dimensional pure non-Abelian anomalies (3.4c), (3.4d)

$$\sum_R F_{1/2}(R) B_R = 0, \quad (C.18)$$

$$\sum_R F_{1/2}(R) C_R = -3 \frac{b^\alpha}{\lambda(\mathfrak{g})} \frac{b^\beta}{\lambda(\mathfrak{g})} \Omega_{\alpha\beta},$$

where the constants  $B_R, C_R$  are defined as

$$\text{tr}_R \hat{F}^4 = B_R \text{tr}_f \hat{F}^4 + C_R (\text{tr}_f \hat{F}^2)^2. \quad (C.19)$$

Expanding the traces on both sides of (C.19) and taking derivatives with respect to  $F^I$  we obtain in analogy to (C.11)

$$\begin{aligned} & \sum_{w \in R} w_I w_J w_K w_L \\ &= B_R \sum_{w^f} w_I^f w_J^f w_K^f w_L^f + \frac{1}{3} C_R \left[ \left( \sum_{w^f} w_I^f w_J^f \right) \left( \sum_{w'^f} w_K'^f w_L'^f \right) + \left( \sum_{w^f} w_I^f w_K^f \right) \left( \sum_{w'^f} w_J'^f w_L'^f \right) \right. \\ & \quad \left. + \left( \sum_{w^f} w_I^f w_L^f \right) \left( \sum_{w'^f} w_J'^f w_K'^f \right) \right]. \end{aligned} \quad (C.20)$$

Like in the preceding subsection we now evaluate explicitly the sums over the fundamental weights in order to rewrite (C.20). For the different simple Lie algebras and all possible choices of indices (C.20) then becomes:

1.  $\mathbf{A}_n$

(a)  $\underline{I = J = K = L}$

$$\sum_{w \in R} (w_I)^4 = B_R \mathcal{C}_{II} \lambda(\mathfrak{g}) + C_R \mathcal{C}_{II}^2 \lambda(\mathfrak{g})^2, \quad (C.21)$$

(b)  $\underline{I = K = L, I \neq J}$

$$\sum_{w \in R} (w_I)^3 w_J = B_R \mathcal{C}_{IJ} \lambda(\mathfrak{g}) + C_R \mathcal{C}_{II} \mathcal{C}_{IJ} \lambda(\mathfrak{g})^2, \quad (C.22)$$

(c)  $\underline{I = L, I \neq J \neq K}$

$$\sum_{w \in R} (w_I)^2 w_J w_K = \frac{1}{3} C_R (2 \mathcal{C}_{IJ} \mathcal{C}_{IK} + \mathcal{C}_{II} \mathcal{C}_{JK}) \lambda(\mathfrak{g})^2, \quad (C.23)$$

(d)  $\underline{I \neq J \neq K \neq L}$

$$\sum_{w \in R} w_I w_J w_K w_L = \frac{1}{3} C_R (\mathcal{C}_{IJ} \mathcal{C}_{KL} + \mathcal{C}_{IK} \mathcal{C}_{JL} + \mathcal{C}_{IL} \mathcal{C}_{JK}) \lambda(\mathfrak{g})^2. \quad (C.24)$$

We can now insert these equations into the Chern-Simons matching (C.17) and find two linearly independent equations

$$\sum_R F_{1/2}(R) \left( \frac{1}{2} B_R + C_R \right) = -3 \frac{b^\alpha}{\lambda(\mathfrak{g})} \frac{b^\beta}{\lambda(\mathfrak{g})} \Omega_{\alpha\beta}, \quad (\text{C.25})$$

$$\sum_R F_{1/2}(R) C_R = -3 \frac{b^\alpha}{\lambda(\mathfrak{g})} \frac{b^\beta}{\lambda(\mathfrak{g})} \Omega_{\alpha\beta}.$$

These equations are in fact equivalent to the gauge anomaly conditions (C.18).

## 2. $\mathbf{B_n}$

(a)  $I = J = K = L$

$$\sum_{w \in R} (w_I)^4 = \frac{1}{4} B_R \mathcal{C}_{In}^2 \mathcal{C}_{II} \lambda(\mathfrak{g}) + C_R \mathcal{C}_{II}^2 \lambda(\mathfrak{g})^2, \quad (\text{C.26})$$

(b)  $I = K = L, I \neq J$

$$\sum_{w \in R} (w_I)^3 w_J = \frac{1}{4} B_R \mathcal{C}_{In}^2 \mathcal{C}_{IJ} \lambda(\mathfrak{g}) + C_R \mathcal{C}_{II} \mathcal{C}_{IJ} \lambda(\mathfrak{g})^2, \quad (\text{C.27})$$

(c)  $I = L, I \neq J \neq K$

$$\sum_{w \in R} (w_I)^2 w_J w_K = \frac{1}{3} C_R (2 \mathcal{C}_{IJ} \mathcal{C}_{IK} + \mathcal{C}_{II} \mathcal{C}_{JK}) \lambda(\mathfrak{g})^2, \quad (\text{C.28})$$

(d)  $I \neq J \neq K \neq L$

$$\sum_{w \in R} w_I w_J w_K w_L = \frac{1}{3} C_R (\mathcal{C}_{IJ} \mathcal{C}_{KL} + \mathcal{C}_{IK} \mathcal{C}_{JL} + \mathcal{C}_{IL} \mathcal{C}_{JK}) \lambda(\mathfrak{g})^2. \quad (\text{C.29})$$

Insertion into (C.17) yields

$$\sum_R F_{1/2}(R) \left( \frac{1}{4} B_R + C_R \right) = -3 \frac{b^\alpha}{\lambda(\mathfrak{g})} \frac{b^\beta}{\lambda(\mathfrak{g})} \Omega_{\alpha\beta}, \quad (\text{C.30})$$

$$\sum_R F_{1/2}(R) C_R = -3 \frac{b^\alpha}{\lambda(\mathfrak{g})} \frac{b^\beta}{\lambda(\mathfrak{g})} \Omega_{\alpha\beta},$$

which is equivalent to (C.18).

## 3. $\mathbf{C_n}$

(a)  $I = J = K = L$

$$\sum_{w \in R} (w_I)^4 = B_R \mathcal{C}_{II} \lambda(\mathfrak{g}) + C_R \mathcal{C}_{II}^2 \lambda(\mathfrak{g})^2, \quad (\text{C.31})$$

(b)  $I = K = L, I \neq J$

$$\sum_{w \in R} (w_I)^3 w_J = B_R \mathcal{C}_{IJ} \lambda(\mathfrak{g}) + C_R \mathcal{C}_{II} \mathcal{C}_{IJ} \lambda(\mathfrak{g})^2, \quad (\text{C.32})$$

(c)  $I = L, I \neq J \neq K$

$$\sum_{w \in R} (w_I)^2 w_J w_K = \frac{1}{3} C_R (2 \mathcal{C}_{IJ} \mathcal{C}_{IK} + \mathcal{C}_{II} \mathcal{C}_{JK}) \lambda(\mathfrak{g})^2, \quad (\text{C.33})$$

(d)  $I \neq J \neq K \neq L$

$$\sum_{w \in R} w_I w_J w_K w_L = \frac{1}{3} C_R (\mathcal{C}_{IJ} \mathcal{C}_{KL} + \mathcal{C}_{IK} \mathcal{C}_{JL} + \mathcal{C}_{IL} \mathcal{C}_{JK}) \lambda(\mathfrak{g})^2, \quad (\text{C.34})$$

which can be inserted into (C.17)

$$\begin{aligned} \sum_R F_{1/2}(R) \left( \frac{1}{4} B_R + C_R \right) &= -3 \frac{b^\alpha}{\lambda(\mathfrak{g})} \frac{b^\beta}{\lambda(\mathfrak{g})} \Omega_{\alpha\beta}, \\ \sum_R F_{1/2}(R) C_R &= -3 \frac{b^\alpha}{\lambda(\mathfrak{g})} \frac{b^\beta}{\lambda(\mathfrak{g})} \Omega_{\alpha\beta}. \end{aligned} \quad (\text{C.35})$$

These equations are equivalent to the anomaly conditions (C.18).

#### 4. $\mathbf{D_n}$

(a)  $I = J = K = L$

$$\sum_{w \in R} (w_I)^4 = B_R \mathcal{C}_{II} \lambda(\mathfrak{g}) + C_R \mathcal{C}_{II}^2 \lambda(\mathfrak{g})^2, \quad (\text{C.36})$$

(b)  $I = K = L, I \neq J$

$$\sum_{w \in R} (w_I)^3 w_J = B_R \mathcal{C}_{IJ} \lambda(\mathfrak{g}) + C_R \mathcal{C}_{II} \mathcal{C}_{IJ} \lambda(\mathfrak{g})^2, \quad (\text{C.37})$$

(c)  $I = L, I \neq J \neq K$

$$\sum_{w \in R} (w_I)^2 w_J w_K = \alpha_{IJK} B_R + \frac{1}{3} C_R (2 \mathcal{C}_{IJ} \mathcal{C}_{IK} + \mathcal{C}_{II} \mathcal{C}_{JK}) \lambda(\mathfrak{g})^2, \quad (\text{C.38})$$

(d)  $I \neq J \neq K \neq L$

$$\sum_{w \in R} w_I w_J w_K w_L = \frac{1}{3} C_R (\mathcal{C}_{IJ} \mathcal{C}_{KL} + \mathcal{C}_{IK} \mathcal{C}_{JL} + \mathcal{C}_{IL} \mathcal{C}_{JK}) \lambda(\mathfrak{g})^2, \quad (\text{C.39})$$

with the definition

$$\alpha_{IJK} := 4 \left( \delta_{I,n-2} \delta_{(J,n} \delta_{K),n-1} - \delta_{I,n-1} \delta_{(J,n} \delta_{K),n-2} - \delta_{I,n} \delta_{(J,n-1} \delta_{K),n-2} \right) \quad (\text{C.40})$$

Inserting into (C.17) we obtain

$$\sum_R F_{1/2}(R) \left( \frac{1}{4} B_R + C_R \right) = -3 \frac{b^\alpha}{\lambda(\mathfrak{g})} \frac{b^\beta}{\lambda(\mathfrak{g})} \Omega_{\alpha\beta}, \quad (\text{C.41})$$

$$\sum_R F_{1/2}(R) C_R = -3 \frac{b^\alpha}{\lambda(\mathfrak{g})} \frac{b^\beta}{\lambda(\mathfrak{g})} \Omega_{\alpha\beta},$$

which is equivalent to the anomaly conditions (C.18).

#### 5. $\mathbf{E}_6, \mathbf{E}_7, \mathbf{E}_8, \mathbf{F}_4, \mathbf{G}_2$

For these algebras there is no fourth-order Casimir, therefore by definition  $B_R = 0$  for all representations. We find by explicit calculation

(a)  $\underline{I = J = K = L}$

$$\sum_{w \in R} (w_I)^4 = C_R \mathcal{C}_{II}^2 \lambda(\mathfrak{g})^2 \quad (\text{C.42})$$

(b)  $\underline{I = K = L, I \neq J}$

$$\sum_{w \in R} (w_I)^3 w_J = C_R \mathcal{C}_{II} \mathcal{C}_{IJ} \lambda(\mathfrak{g})^2 \quad (\text{C.43})$$

(c)  $\underline{I = L, I \neq J \neq K}$

$$\sum_{w \in R} (w_I)^2 w_J w_K = \frac{1}{3} C_R (2 \mathcal{C}_{IJ} \mathcal{C}_{IK} + \mathcal{C}_{II} \mathcal{C}_{JK}) \lambda(\mathfrak{g})^2 \quad (\text{C.44})$$

(d)  $\underline{I \neq J \neq K \neq L}$

$$\sum_{w \in R} w_I w_J w_K w_L = \frac{1}{3} C_R (\mathcal{C}_{IJ} \mathcal{C}_{KL} + \mathcal{C}_{IK} \mathcal{C}_{JL} + \mathcal{C}_{IL} \mathcal{C}_{JK}) \lambda(\mathfrak{g})^2 \quad (\text{C.45})$$

Plugging this in into the Chern-Simons matching (C.17) we get

$$\sum_R F_{1/2}(R) C_R = -3 \frac{b^\alpha}{\lambda(\mathfrak{g})} \frac{b^\beta}{\lambda(\mathfrak{g})} \Omega_{\alpha\beta}, \quad (\text{C.46})$$

which is again equivalent to the cancelation of anomalies since the first equation in (C.18) is trivial due to the absence of a fourth-order Casimir.

Thus we have shown that the matching condition from one-loop Chern-Simons terms (C.17) is fully equivalent to the cancelation of non-Abelian gauge anomalies (C.18) for all simple Lie algebras.

## D Intersection numbers and matchings

In this section we list useful intersection numbers of elliptically fibered Calabi-Yau four- and threefolds along with their matched quantity in the M-/F-theory duality. The definitions of the indices are as in section 4.

For Calabi-Yau fourfolds we consider

$$\Theta_{\Lambda\Sigma} = -\frac{1}{4} D_{\Lambda} \cdot D_{\Sigma} \cdot G_4, \quad \mathcal{K}_{\Lambda\Sigma}{}^{\alpha} = \eta^{-1}{}_{\beta}{}^{\alpha} D_{\Lambda} \cdot D_{\Sigma} \cdot \mathcal{C}^{\beta}, \quad (\text{D.1})$$

where  $\eta_{\alpha}{}^{\beta}$  is the full-rank intersection matrix (4.1). One finds

$$\begin{aligned} \Theta_{\alpha\beta} &= 0, & \Theta_{\alpha 0} &= 0, & \Theta_{\alpha I} &= 0, & \Theta_{\alpha m} &= \frac{1}{2} \theta_{\alpha m}, \\ \mathcal{K}_{\alpha\beta}{}^{\gamma} &= 0, & \mathcal{K}_{0\alpha}{}^{\beta} &= \delta_{\alpha}^{\beta}, & \mathcal{K}_{m\alpha}{}^{\beta} &= 0, & \mathcal{K}_{I\alpha}{}^{\beta} &= 0, \\ \mathcal{K}_{00}{}^{\alpha} &= 0, & \mathcal{K}_{0m}{}^{\alpha} &= 0, & \mathcal{K}_{0I}{}^{\alpha} &= 0, & \mathcal{K}_{mn}{}^{\alpha} &= -b_{mn}^{\alpha}, \\ & & & & & & \mathcal{K}_{IJ}{}^{\alpha} &= -b^{\alpha} \mathcal{C}_{IJ}. \end{aligned} \quad (\text{D.2})$$

Finally for Calabi-Yau threefolds there are intersection numbers

$$k_{\Lambda\Sigma\Theta} = D_{\Lambda} \cdot D_{\Sigma} \cdot D_{\Theta}, \quad k_{\Lambda} = D_{\Lambda} \cdot c_2, \quad (\text{D.3})$$

and one evaluates

$$\begin{aligned} k_{\alpha\beta\gamma} &= 0, & k_{0\alpha\beta} &= \Omega_{\alpha\beta}, & k_{I\alpha\beta} &= 0, \\ k_{m\alpha\beta} &= 0, & k_{00\alpha} &= 0, & k_{IJ\alpha} &= -\Omega_{\alpha\beta} b^{\beta} \mathcal{C}_{IJ}, \\ k_{mn\alpha} &= -\Omega_{\alpha\beta} b_{mn}^{\beta}, & k_{0I\alpha} &= 0, & k_{0m\alpha} &= 0, \\ k_{\alpha} &= -12 \Omega_{\alpha\beta} a^{\beta}. \end{aligned} \quad (\text{D.4})$$

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